Traced monoidal categories as algebraic structures in $\mathbf{Prof}$

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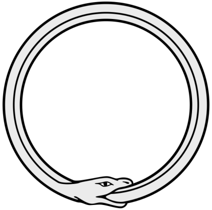
Tracing, recursion, feedback



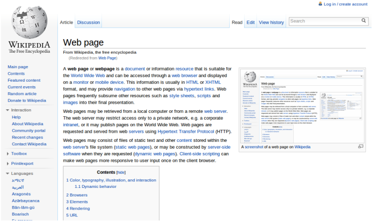
Ouroboros

$$factorial :: Integral\ a \Rightarrow a \rightarrow a$$
$$factorial\ 0 = 1$$
$$factorial\ n = n * factorial\ (n - 1)$$

Haskell `factorial`

$$\frac{}{0 \in \mathbb{N}} \qquad \frac{n \in \mathbb{N}}{\mathrm{succ}(n) \in \mathbb{N}}$$

Inductive definition of $\mathbb{N}$



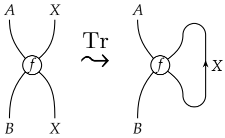
Wikipedia page for ‘Web page’

Traced monoidal categories

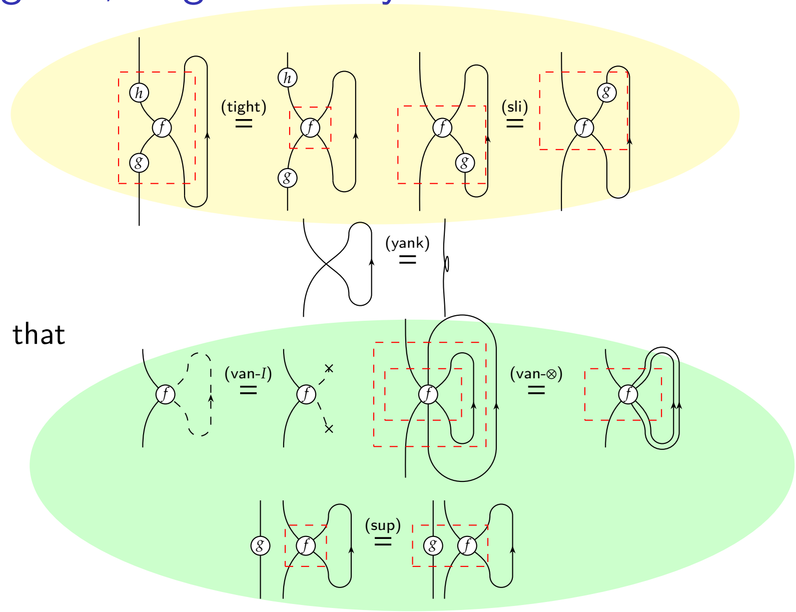
Definition

A traced monoidal category is a structure on a monoidal category, equipping it with an operation which sends each morphism $A \otimes X \xrightarrow{f} B \otimes X$ to its *trace* $A \xrightarrow{\mathrm{Tr}^X_{A,B} f} B$, subject to some extra conditions.

Traced monoidal categories, diagramatically



such that



$\mathbf{Prof}$: compact closed bicategory of profunctors

Definition

A profunctor $\mathscr{C} \overset{P}{\nrightarrow} \mathscr{D}$ is given by the data of a functor $\mathscr{D}^{\mathsf{op}} \times \mathscr{C} \to \mathbf{Set}$.

Definition

$\mathbf{Prof}$ is the compact closed bicategory given by:

0-morphisms — categories;

1-morphisms — profunctors;

2-morphisms — natural transformations;

structural morphisms — Hom (pro)functors.

Internal vs external

Definition (Internal)

A monoid is a set X equipped with an associative unital binary operation $X \times X \xrightarrow{m} X$.

$\xleftarrow[\textbf{Set}]{\text{interpret in}}$

Definition (External)

In a monoidal category, a monoid is an object X equipped with morphisms $X \otimes X \xrightarrow{m} X$ and $I \xrightarrow{u} X$, satisfying associativity and unitality.

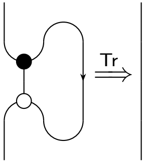
A monoid internal to...

Vect — unital algebra;

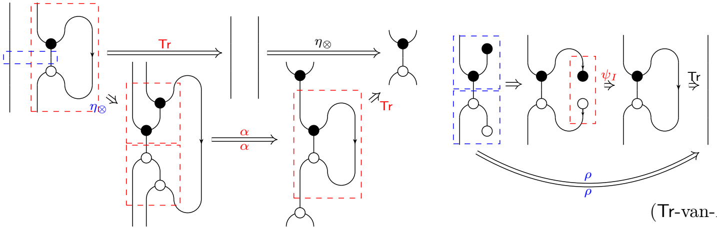
Cat — monoidal category;

Prof — promonoidal category.

External traced monoidal categories

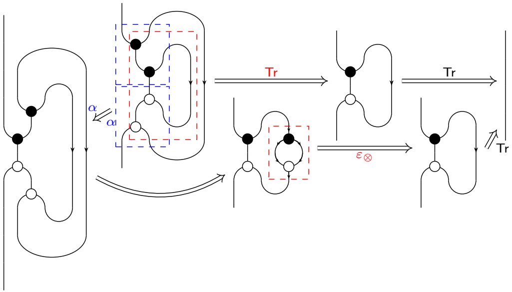


such that



(Tr-sup)

(Tr-van-I)



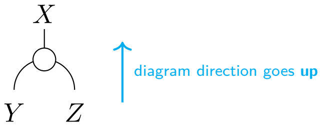
(Tr-van-$\otimes$)

How we proved it

Internal string diagrams I

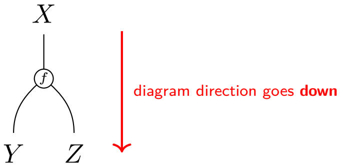
Prof string diagrams represent Hom-sets:

$$\mathscr{C}(X, Y \otimes Z) \rightsquigarrow$$



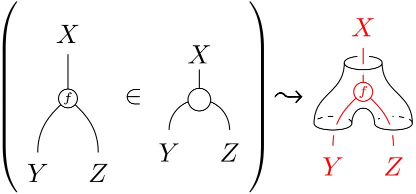
As $\mathscr{C}$ is a monoidal category, it too admits string diagrams:

$$X \xrightarrow{f} Y \otimes Z \rightsquigarrow$$

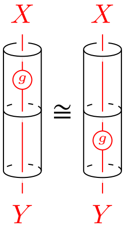


Internal string diagrams II

String diagrams of $\mathscr{C}$ exist in the space obtained by *inflating* the $\mathbf{Prof}$ string diagrams:

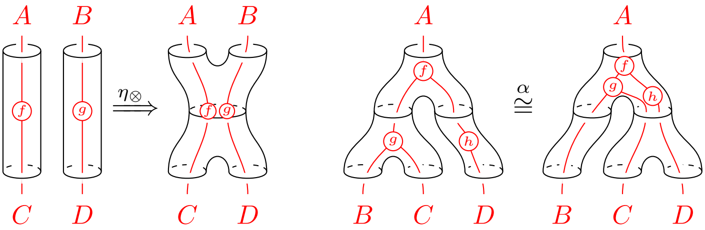


Boundaries between composite $\mathbf{Prof}$ 1-morphisms are elidable:



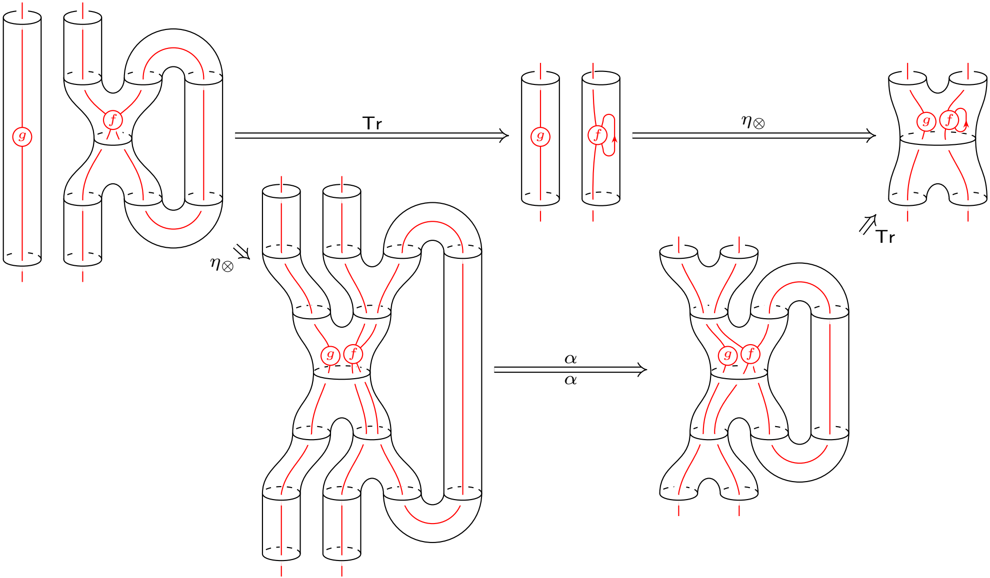
Internal string diagrams III

Prof 2-morphisms are tube-boundary-preserving rewrites which *act* on the internal strings:



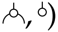
Structural 2-morphisms have *structural* actions

External traced monoidal categories, with internal string diagrams



How we used it

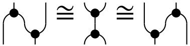
$*$-autonomous categories, externally

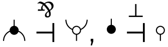
Pseudomonoid basic data ($\mathscr{C}$, ) …plus 2-morphisms and equations making this into a monoid

freely add right adjoints 

Traced pseudomonoid as above, but additionally the Tr 2-morphism …and associated equations;

Frobenius pseudomonoid basic data ($\mathscr{C}$, ) …plus 2-morphisms and equations making this into a **Frobenius** monoid



freely add right adjoints 

derive left adjoints 

Compact closed categories, externally

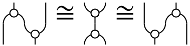
Insight: a compact closed category is a **degenerate** $*$-autonomous category; one in which $\parr$ coincides with $\otimes$

$$\implies \text{white} = \text{black}$$

This seems to suggest that geometrically, the white structure should be Frobenius too!

We have a candidate, which crucially uses the Tr 2-morphism.

Conjecture



If this is true, we are able to formally show that *tracing* combined with *$*$-autonomous* entails *compactness*, **without symmetry**.