Coherent invertibility in associative *n*-categories

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Higher string diagrams

Variants of string diagrams provide diagrammatic calculi for various categorical structures:

Higher string diagrams (contd.)

Not degenerate when regions convey information:

n-dimensional manifold diagrams are diagrams for globular n-categories

semantics	finitely presented n-categories
model	associative n-categories
combinatorial model/syntax	zigzag categories

Finitely-presented globular n-categories

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Also: every ω-category is ‘equivalent’ to a free one.

Example: monoid object in a monoidal category

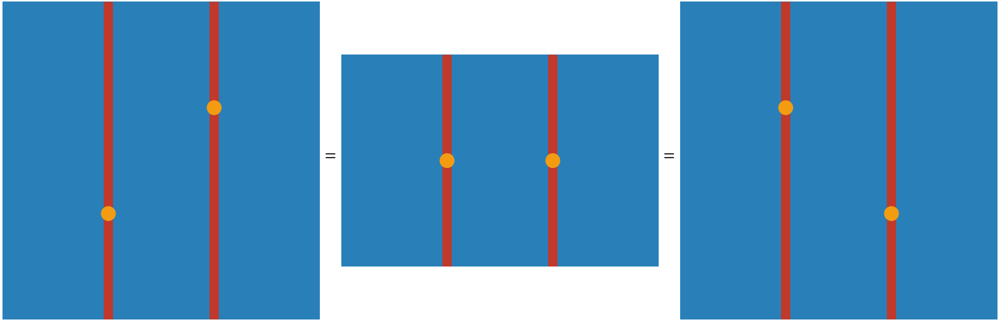
Consider a signature given by:

$$\Sigma_0 := \{x\}$$
$$\Sigma_1 := \{f : x \to x\}$$
$$\Sigma_2 := \{m \colon f \circ f \Rightarrow f, u \colon \mathrm{id}_x \Rightarrow f\}$$
$$\Sigma_3 := \{ \text{ m is associative, u the unit of m } \}$$

Example: monoid object in a monoidal category (contd.)

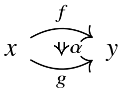
Structural equalities are captured by planar isotopy. E.g. for $e\colon f \Rightarrow f$:

$$\mathrm{id}_x \circ e \cdot e \circ \mathrm{id}_x = e \circ e = e \circ \mathrm{id}_x \cdot \mathrm{id}_x \circ e.$$

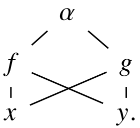


Zigzag categories: motivation

For some signature Σ, let P_Σ denote its face poset. E.g. for a signature Σ with a 2-cell



its face poset is



Zigzag categories: motivation (contd.)

As a category, P_Σ is a 'space' of 0-dimensional diagrams with respect to Σ:

- x and y are objects; ✓
- however, f, g, α are also objects. ✗

What category is the 'space' of the 1-dimensional or 2-dimensional diagrams?

Zigzag categories: definition

Let $\mathcal{C}$ be a category.

$\mathrm{Zig}(\mathcal{C})$ is the category of

objects	iterated cospans of $\mathcal{C}$; feet are *regular* levels, tips are *singular* levels;
morphisms	a collection of $\mathcal{C}$-morphisms arranging into a monotone map between singular levels such that, combined with the dual monotone map between regular levels in the other direction labelled by morphisms of $\mathcal{C}$, the resulting planar diagram commutes in $\mathcal{C}$.

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Let $\mathrm{Zig}^n(\mathcal{C})$ denote the n-fold iterated zigzag category of $\mathcal{C}$:

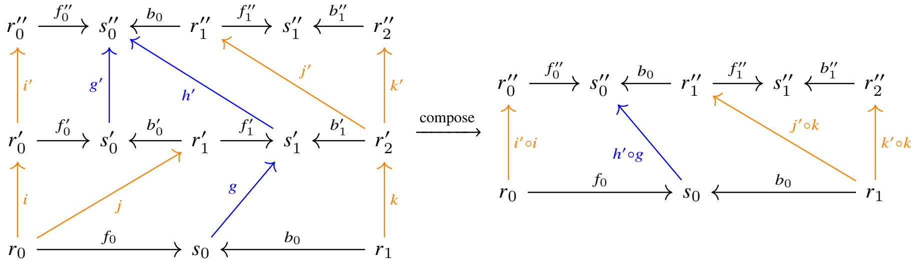
$$\begin{aligned} \mathrm{Zig}^0(\mathcal{C}) &:= \mathcal{C}, \\ \mathrm{Zig}^{n+1}(\mathcal{C}) &:= \mathrm{Zig}(\mathrm{Zig}^n(\mathcal{C})). \end{aligned}$$

Zigzag categories: example

Example:

object $r_0 \xrightarrow{f_0} s_0 \xleftarrow{b_0} r_1 \xrightarrow{f_1} s_1 \xleftarrow{b_1} r_2$

morphism



Zigzag categories: fancy definition I

Let Z be the category with objects $\{r_i^n \mid i \leq n \in \mathbb{N}\} \cup \{s_i^n \mid i < n \in \mathbb{N}\}$, and morphisms sets of monotone maps:

$$\begin{aligned}
\mathsf{Z}(s_i^n, s_j^m) &:= \left\{[n-1] \xrightarrow{\alpha} [m-1] \mid \alpha(i) = j\right\}, \\
\mathsf{Z}(r_i^n, r_j^m) &:= \left\{[n-1] \xrightarrow{\alpha} [m-1] \mid \mathsf{R}\alpha(j) = i\right\}, \\
\mathsf{Z}(r_i^n, s_j^m) &:= \left\{[n-1] \xrightarrow{\alpha} [m-1] \mid \mathsf{R}\alpha(j) \leq i \leq \mathsf{R}\alpha(j+1)\right\}, \\
\mathsf{Z}(s_i^n, r_j^m) &:= \emptyset.
\end{aligned}$$

Composition is given by composition of monotone maps.

There is a canonical functor $\mathsf{Z} \xrightarrow{p} \Delta$ that sends r_i^n and s_i^n to $[n-1]$, the universal zigzag bundle, which is exponentiable in **Cat**.

Zigzag categories: fancy definition II

$\mathbf{Cat} \xrightarrow{\mathrm{Zig}(-)} \mathbf{Cat}$ is the polynomial functor determined by

$$\mathbf{Cat} \cong \mathbf{Cat}/1 \xrightarrow{!_Z{}^*} \mathbf{Cat}/\mathrm{Z} \xrightarrow{\Pi_p} \mathbf{Cat}/\Delta \xrightarrow{\Sigma_{!_\Delta}} \mathbf{Cat}/1 \cong \mathbf{Cat}.$$

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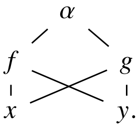
Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Functors $\mathcal{D} \to \mathrm{Zig}(\mathcal{C})$ are in bijection commuting diagrams of the form

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \mathsf{Z} \times_{\boldsymbol{\Delta}} \mathcal{D} & \longrightarrow & \mathsf{Z} \\ & & \downarrow & \lrcorner & \downarrow \\ & & \mathcal{D} & \longrightarrow & \boldsymbol{\Delta}. \end{array}$$

n-dimensional diagrams live in $\mathrm{Zig}^n(P_\Sigma)$

Recall the face poset P_Σ:



- $\mathrm{Zig}(P_\Sigma)$ has objects

$$x \to f \leftarrow y,$$
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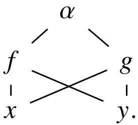

- $\mathrm{Zig}(P_\Sigma)$ has objects

$$\begin{gathered} x \to f \leftarrow y, \\ x \to g \leftarrow y, \\ x \to x \leftarrow x, \\ x \to f \leftarrow y \to y \leftarrow y, \\ \dots \end{gathered}$$

- $\mathrm{Zig}^2(P_\Sigma)$ has objects

$$\begin{array}{ccccc} x & \to & g & \leftarrow & y \\ \downarrow & \searrow & \downarrow & \swarrow & \downarrow \\ x & \to & \alpha & \leftarrow & y \\ \uparrow & \nearrow & \uparrow & \nwarrow & \uparrow \\ x & \to & f & \leftarrow & y, \end{array}$$

$\dots$

Colimits in zigzag categories build 'homotopies'

Height 2 diagram in $\mathrm{Zig}^n(\mathcal{C})$:

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When can we universally shrink this?

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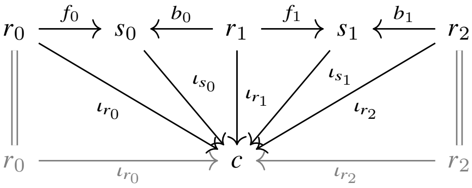

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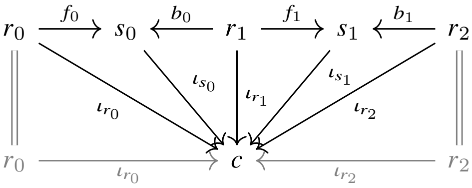

In short: the functor $\mathrm{Zig}^n(\mathcal{C}) \to \Delta$ is (*) a Grothendieck opfibration — compute colimits in Δ and $\mathrm{Zig}^{n-1}(\mathcal{C})$ and lift them to get colimits in $\mathrm{Zig}^n(\mathcal{C})$.

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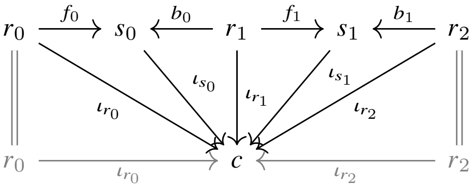

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For more details, see High-level methods for homotopy construction in associative *n*-categories.

Coherence problem for invertibility

Let $x \xrightarrow{f} y$ and $y \xrightarrow{f^{-1}} x$ witness an isomorphism $x \cong y$.

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In a higher category, these give 2-cells:

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Coherence: the sequence

$$f = \underline{\mathrm{id}_y} \circ f \Rightarrow (f \circ f^{-1}) \circ f = f \circ \underline{(f^{-1} \circ f)} \Rightarrow f \circ \mathrm{id}_x = f$$

should correspond to the 'do nothing'/identity 2-cell id_f.

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This itself is a 3-cell, generating further coherence conditions ad infinitum...

Representing inverses

The 1-cell $x \xrightarrow{f} y$ is represented by the object in $\mathrm{Zig}(P_\Sigma)$:

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What about for an endomorphism $x \xrightarrow{g} x$?

Entrance path **Pos**-category I

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$$\begin{array}{c} g \\ | \\ x. \end{array}$$

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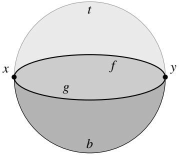
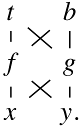
Have forgotten all the information about *where* x is on the boundary of g.

Entrance path **Pos**-category II



Face poset:



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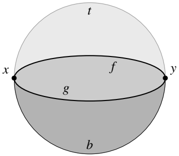

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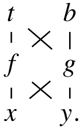

Entrance path **Pos**-category refines the face poset to remember this information:

$$\mathrm{Hom}(x, t) = \{x \to f \to t \Rightarrow x \to t \Leftarrow x \to g \to t\}\,.$$

Pos-enrich everything

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Need:

- a notion of 'framed' zigzag category **Pos**-category;
- what's a **Pos**-enriched colimit in this context?
- when does an object of $\mathrm{Zig}^n(P_\Sigma)$ represent an n-dimensional diagram?

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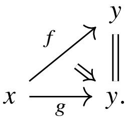
Example:



Framed zigzag **Pos**-categories

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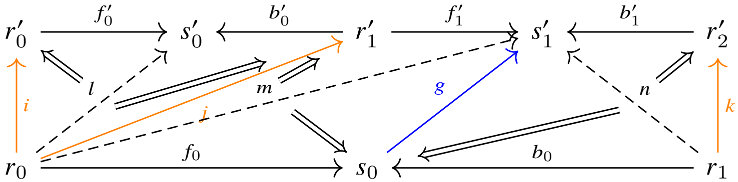
morphisms: a collection of $\mathcal{C}$-morphisms arranging into a monotone map between singular levels such that, combined with the dual monotone map between regular levels in the other direction labelled by morphisms of $\mathcal{C}$, the resulting planar diagram **oplaxly** commutes in $\mathcal{C}$;

2-cell fillers: pointwise inherited from $\mathcal{C}$ – let $Z \overset{f}{\underset{g}{\rightrightarrows}} Z'$ be morphisms; $f \Rightarrow g$ if and only if f and g have the same underlying monotone maps, and f is below g with respect to these components in $\mathcal{C}$.

Framed zigzag **Pos**-categories: example



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- if I has a terminal object 1, this may **not** determine an oplax conical colimit.

Oplax conical colimits are pathological

Let $\mathcal{D}$ be the **Pos**-category generated by:

$$x \underset{f}{\overset{g}{\underset{\Uparrow}{\rightrightarrows}}} y;$$

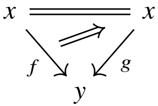
and F and F' given respectively by diagrams:

$$x \qquad\qquad\qquad x = x.$$

F has both an oplax conical colimit and a conical colimit, given by the identity cocone with tip x and leg given by id_x.

Oplax conical colimits are pathological (contd.)

However, the identity cocone over F' is not an oplax conical colimit: the oplax cocone



exists; if the identity cocone were universal, then this oplax cocone would factor through it, which is tantamount to requiring a morphism $x \to y$ which is simultaneously equal to f and g, which does not exist.

Weakening to quasi colimits

Consider the universal properties:

- colimits in (**Set**-enriched) categories

$$\forall c \in \mathcal{C}.\mathcal{C}(\mathrm{colim}\, F, c) \cong \mathbf{Cocone}(F, c);$$

- oplax conical colimits in **Pos**-enriched categories

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Idea: replace the order isomorphism with an adjunction of posets.

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- oplax conical colimits in **Pos**-enriched categories

$$\forall c \in \mathcal{C}. \mathcal{C}(\mathrm{oplax\ colim}\, F, c) \cong \mathbf{OplaxCocone}(F, c).$$

Sets are either isomorphic or not, but posets admit weaker notions of 'isomorphism': (monotone) Galois connections.

Idea: replace the order isomorphism with an adjunction of posets.

These are oplax **quasi** colimits.

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Theorem: an oplax rari colimit exists when the diagram index admits a right-adjoint quasi terminal object.

Topos of reflexive directed graphs

Let $\mathcal{Q}$ denote the category generated by

$$V \underset{t}{\overset{s}{\underset{\leftarrow r -}{\rightrightarrows}}} E,$$

where r is a common retract of s and s, i.e. $r \circ s = \mathrm{id}_V = r \circ t$.

The topos of reflexive directed graphs is the category of presheaves (functors $\mathcal{Q}^{\mathrm{op}} \to \mathbf{Set}$):

$$\mathbf{Graph} := \hat{\mathcal{Q}}.$$

Presheaf topos of W-weighted graphs I

A graph weighting W is a collection of vertex weights W_v, and edge weights W_e, such that edge labels have both a source and target vertex label associated to them, and vertex labels have an associated self-loop edge label.

We write $f : a \to b \in W_e$ to denote that the edge label f has source vertex label $a \in W_v$ and target vertex label $b \in W_v$, and $r_a \in W_e$ to denote the distinguished self-loop edge label associated to the vertex label $a \in W_v$.

Presheaf topos of W-weighted graphs II

Let $\mathcal{C}^{\text{op}} \xrightarrow{X} \textbf{Set}$ be a presheaf in $\hat{\mathcal{C}}$. There is an equivalence of categories

$$\hat{\mathcal{C}}/X \cong \widehat{\int X},$$

where $\int X$ denotes the category of elements of X.

Presheaf topos of W-weighted graphs III

Let $\mathcal{Q}^{\mathrm{op}} \xrightarrow{W} \mathbf{Set}$ be a graph weighting, presented as a reflexive directed graph.

$\int W$ is the category with

objects
- $a, b, c, \ldots \in W_v$ for each vertex weight;
- $\alpha, \beta, \gamma, \ldots, r_a, r_b, r_c, \ldots \in W_e$ for each edge weight;

morphisms
- a 'typed source' morphism $a \xrightarrow{s_\alpha} \alpha$ whenever $\alpha \colon a \to b$ for any b;
- a 'typed target' morphism $b \xrightarrow{t_\alpha} \alpha$ whenever $\alpha \colon a \to b$ for any a;
- a 'typed reflexive' morphism $r_a \xrightarrow{r_a} a$ for any a.

Homotopy theory of W-weighted graphs I

Graph homomorphisms $G \to H$ correspond to global elements of H^G, i.e. morphisms $W \to H^G$.

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For $G \underset{g}{\overset{f}{\rightrightarrows}} H$, homotopy $f \overset{h}{\simeq} g$ should be a 'path' $I_n \to H^G$ from f to g for some $n \in \mathbb{N}$.

By currying, this is equivalently a graph homomorphism $G \times I_n \to H$.

Homotopy theory of W-weighted graphs II

Let $G \overset{f}{\underset{g}{\rightrightarrows}} H$ be two parallel homomorphisms between W-weighted reflexive directed graphs.

An n-step homotopy $G \times I_n \xrightarrow{h} H$ from f to g is a W-weighted reflexive directed graph homomorphism, for some $I_n \in \mathcal{I}_n$, such that

$$\begin{aligned}
\forall w \in W_v.\forall v \in Gw.h(v, w_0) &= f(v), \\
h(v, w_n) &= g(v), \\
\forall w \in W_e.\forall e \in Ge.h(e, w_0) &= f(e), \\
h(e, w_n) &= g(e),
\end{aligned}$$

where w_i represents the weight vertex/edge of the ith copy of w in I_n.

We say that f and g are homotopic, and write $f \overset{h}{\simeq} g$, if there exists an n-step homotopy h between them some $n \in \mathbb{N}$.

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- there is a degenerate edge $u \to v$ or $u \leftarrow v$.

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An object in a framed zigzag category represents an n-dimensional diagram when its stiff representative is isomorphic to one built from its signature.

Thanks for listening!

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