Invertibility in higher-categorical string diagrams with framed zigzags

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Abstract

Zigzag categories have been employed to give a combinatorial model of directed higher categories amenable to computer implementation, presented as n-dimensional string diagrams. However, the theory lacks a representation of inverse, which would additionally require the existence of cancellation moves along with an infinite collection of higher-dimensional coherence data.

We generalise the theory to support invertibility by equipping generators with *framing* data, allowing for a natural representation of inverse. Using this *framed zigzags* approach, we show that the required cancellation moves and coherence data arise uniformly via a colimit mechanism. We use tools from enriched category theory to justify correctness of our constructions, and exhibit a proof assistant which implements our theory.

CCS Concepts

• **Theory of computation** → **Categorical semantics**; *Rewrite systems*; *Automated reasoning*; • **Mathematics of computing** → **Algebraic topology**.

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1 Introduction

n-dimensional string diagrams and invertibility. Reasoning in monoidal categories can be performed in a two-dimensional graphical calculus of string diagrams, which is two-dimensional because monoidal structure allows for the composition of morphisms in two ways: ordinary composition, and tensor product. Every monoidal category can be seen as a one-object bicategory, and string diagrams in general bicategories are obtained by allowing the regions to carry information as colours. Thus, bicategories are the general categorical structure for *bimodal* composition, and string diagrams represent this composition spatially. n-dimensional string diagrams, which compose by juxtaposition n-dimensions, extend this correspondence to weak n-categories[1], specialising to the usual notion of string diagram for $n=2$ and *surface diagrams* for $n=3$.

Zigzag categories combinatorially encode n-dimensional string diagrams as objects of an *n-fold zigzag category* $\mathrm{Zig}^n(C)$, which is a functorial construction over a base category C representing an algebraic signature that finitely presents the higher category. Just as a morphism in a category may be invertible in their sole dimension of composition, n-cells in higher categories may be *invertible* in any combination of their n dimensions. For instance, a 2-cell scalar (Section 4.2) may be *vertically* invertible, *horizontally* invertible, or both, or neither. The spacial nature of composition in n-dimensions also suggests what the representation of these inverses should be: the same string diagram, but *mirrored* in the axes of composition for which it is invertible. In the existing zigzag framework, there is no way to represent such invertible generators at all.

Coherent equivalences. In a category, a morphism $f: x \to y$ is invertible if it admits an inverse $f^{-1}: y \to x$, such that the composition in either order is the identity morphism:

$$f^{-1} \circ f = \mathrm{id}_x, \qquad f \circ f^{-1} = \mathrm{id}_y.$$

In this case, we say that f and f^{-1} are an inverse pair, witnessing the isomorphism $x \cong y$. In a higher category, this structure naturally generalises to that of an *equivalence*, where the equalities above are replaced by directed 2-cells that perform pair cancellations or introductions:

$$f^{-1} \circ f \Rightarrow \mathrm{id}_x, \qquad f \circ f^{-1} \Rightarrow \mathrm{id}_y,$$
$$f^{-1} \circ f \Leftarrow \mathrm{id}_x, \qquad f \circ f^{-1} \Leftarrow \mathrm{id}_y.$$

We may require these 2-cells to themselves form equivalences as inverse pairs. This yields a coinductive definition of equivalence, which naively generates this structure in all dimensions [21].

The resulting notion of equivalence has poor algebraic properties. To see why, consider the following 2-cell, obtained as a composite of the 2-cells described above:

$$f = f \circ \underline{\mathrm{id}_x} \Rightarrow (f \circ f^{-1}) \circ f = f \circ \underline{(f^{-1} \circ f)} \Rightarrow \mathrm{id}_y \circ f = f. \tag{1}$$

Here we underline the redex applying in each case, and assume for simplicity our theory is definitionally unital and associative. This 2-cell is called the *snake composite*, and has type $f \Rightarrow f$. We might expect it to be equivalent to id_f: this must certainly be the case if we desire the free ∞-category generated by an equivalence $f: x \to y$ to itself be equivalent to the free ∞-category generated by a single object, where parallel morphisms are always equivalent. However, under the coinductive definition of equivalence presented before, this will not be the case; we say that this notion of equivalence is *incoherent*.

For a coherent theory of equivalences, the snake composite must therefore be equivalent to the identity (via a coherent equivalence). But this is not the end of the story: there is an infinite sequence of such phenomena, known as *catastrophes* [4] from their study in manifold theory, with one qualitatively new example arising in each dimension[2]. Any computer algebra system for higher categories must have a solution to this problem in order to implement a satisfactory theory of equivalences.

[1] Specifically, in the model of *associative n-categories* [7].

[2] The first few have enigmatic names: the snake, swallowtail, butterfly and wigwam.

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String diagrams. This paper presents a solution to this representation problem, which seems to generate all coherent equivalences, for the theory of string diagrams. We work with the formalism of *zigzag categories* [20], which has been developed into a proof assistant homotopy.io [20, 11, 25]. This proof assistant encodes string diagrams combinatorially using a simple inductive 1-categorical structure called a *zigzag*. It allows the construction of composite higher morphisms in a finitely generated higher category, using a geometrical string diagram interface. Given the difficulty of encoding coherent equivalences as discussed above, until this work, all the generators of the theory were directed (not invertible).

The main technique for introducing nontrivial structure in the proof assistant is *contraction*, which uses a colimit operation to simplify some part of a diagram [20]. The simplest nontrivial example looks like Figure 1, where we collapse a diagram in which two 2-cells are composed at different heights. This contraction procedure can be applied both in the top dimension as demonstrated here, and also recursively within subdiagrams, to allow the construction of complex proofs.

In this article, we show how the technology of zigzags and contraction can be modified to allow the definition of invertible generators and manipulation of coherent inverses. As a result, the proof assistant homotopy.io gains the ability to work with these, and hence becomes a system for manipulating finitely presented higher *groupoids*. This new capability will be enabled by introduction of *framing information* at the level of zigzags.

To understand this, we consider a 1-cell $e: x \rightarrow x$. In the zigzag formalism, this is encoded combinatorially as the following cospan:

$$x \rightarrow e \leftarrow x.$$

Suppose we would like e to be invertible. Then how can we encode e^{-1}? One approach may be to introduce 'e^{-1}' as a formal token, and allow the following zigzag to represent the inverse 1-cell:

$$x \rightarrow e^{-1} \leftarrow x.$$

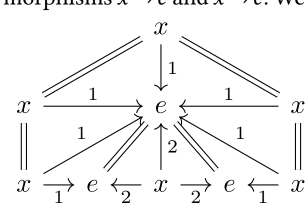
However, this is unsatisfactory: we know that an infinite sequence of higher morphisms will be required to ensure the inverse is coherent — including the snake and its associated higher catastrophes — but their structure is chaotic, and we do not know of a good way to introduce a corresponding syntax of higher tokens.

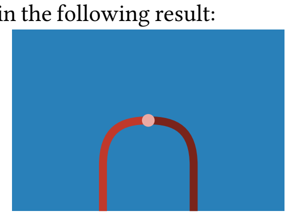
Our solution is to introduce for each arrow $x \rightarrow e$ a *frame*, as a decoration on the arrow. This allows us to distinguish e and e^{-1} as follows:

$$x \xrightarrow{1} e \xleftarrow{2} x \qquad\qquad x \xrightarrow{2} e \xleftarrow{1} x$$

The labels 1 and 2 are formal tokens to distinguish the framings; the choice of label does not have any special meaning. We call this extended structure a *framed zigzag*. This has the attractive property that the framed zigzag representations of e and e^{-1} are mirror-images.

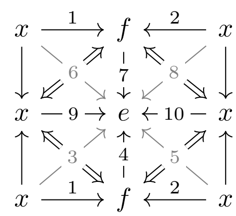
Using this idea, we consider applying the contraction principle to the composite $e \circ e^{-1}$. Contraction is computed by colimit, which is easily computed over our base category with objects $\{x, e\}$ and morphisms $x \xrightarrow{1} e$ and $x \xrightarrow{2} e$. We obtain the following result:

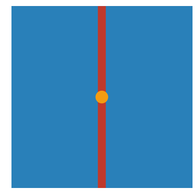




We observe that the contraction procedure performs the cancellation of e with e^{-1} as intended. Above, we append two triangles which close off the geometry, yielding an upper boundary labelled by x. Overall this is the correct geometrical representation of the cancellation move $e \circ e^{-1} \Rightarrow \mathrm{id}_x$, depicted in string diagrams on the right.

Posetal enrichment. We extend the same principle to generators of higher dimension. For example, a 2-cell $f: e \Rightarrow e$ would be represented as follows, where we give both the framed zigzag representation in addition to the geometrical rendering produced by the proof assistant:





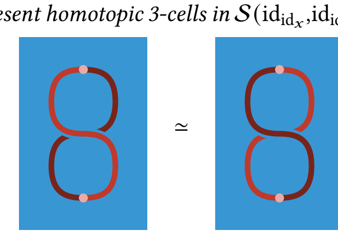
Here we see a further new aspect of our construction: where a triangle appears in the neighbourhood of a generator, we give it a 2-cell filler. This 2-cell structure will be posetal, and so we can handle it with the technology of **Pos**-enriched categories. For any 2-cell generator, its neighbourhood structure is therefore encoded by a particular choice of finite **Pos**-enriched category.

As the dimension goes up, this **Pos**-enriched categorical framework remains constant. Therefore, even for an n-dimensional generator X where $n > 2$, its neighbourhood is completely defined by a choice of **Pos**-category, and the same will be true for all composite geometries. In this way we are able to encode higher categorical diagrams of arbitrary dimension, with a formal foundation that stays at the level of **Pos**-categories. Precedent for this comes from the work of Nanda [18], who uses **Pos**-categories as a categorical setting for developing discrete Morse theory. As a consequence of this **Pos**-enrichment, contraction must now be handled with *marked* colimits.

Implementation. These ideas have been implemented, and are available in a pre-release version of the proof assistant homotopy.io, available at https://beta.homotopy.io.

The invertibility developed in this paper allows for spaces to be presented algebraically as string diagrams, which can then be manipulated in homotopy.io to deduce theorems about topology. Moreover, the topological nature of string diagrams closely aligns with the geometrical intuition of homotopy theory. To demonstrate the application of invertible generators, we illustrate in Appendix D a proof of the following result, along with a link to a video representing the proof as a smooth animation of 3-dimensional structures [13].

Theorem 1.1 (Online: [16]). *In the free ∞-groupoid $\mathcal{S}$ generated by an object x and a 2-cell $f: \mathrm{id}_x \Rightarrow \mathrm{id}_x$, the following 'Figure-8' string diagrams represent homotopic 3-cells in $\mathcal{S}(\mathrm{id}_{\mathrm{id}_x}, \mathrm{id}_{\mathrm{id}_x})$:*



(a) String diagram representation

(b) Zigzag representation

Figure 1: Contraction example

By the homotopy hypothesis, $\mathcal{S}$ generates the homotopy type of the 2-sphere S^2, with homotopy classes of 3-cells in each Hom space in bijection with the elements of the homotopy group $\pi_3(S^2)$.

The Figure-8 diagrams are themselves composites of the invertible structure, along with a braiding move. A priori, it appears there are four such Figure-8 diagrams, coming in two inverse pairs, and therefore we might expect two distinct homotopy generators in this class, yielding $\pi_3(S^2) \cong \mathbb{Z}^2$. However, due to the non-obvious equivalence established in the theorem, in fact these generators are homotopy equivalent, and hence $\pi_3(S^2) \cong \mathbb{Z}$.

It is in this sense that our tool, homotopy.io, is described as a *proof assistant*, facilitating the formalisation and study of these theories.

1.1 Related work

This work is part of a larger programme to develop a proof assistant for higher category theory in the model of associative n-categories [7], homotopy.io [5]. In particular, it builds on the zigzag construction and contraction operation described by Reutter and Vicary [20]. We focus on a high-level categorical analysis of specifically framed zigzags and framed contraction, as opposed to an explicit algorithmic description of the broader system, which is described by Hu [15]. Other important constructions for zigzag categories are given by Heidemann, Reutter and Vicary [11], Sarti and Vicary [22] and Tataru and Vicary [25, 26].

The predecessor tool *Globular* [1] directly encoded the axioms of a quasistrict globular 4-category, and as such is fundamentally limited to dimension 4. It included an *incoherent* notion of inverse, with the coinductive structure of cancellations, but without the snake equation and higher coherence moves.

Existing type-theoretic models of higher groupoids [2, 3], including homotopy type theory [27], handle invertible structures in a coherent way. The type theory CaTT [8] can handle both directed and coherent undirected inverses, although that point is not made very clearly in the literature. The novelty of our approach is to use new techniques that extend support of invertible generators to a geometrical string-diagram setting.

1.2 Outline

The outline of our contribution is as follows. Section 2 introduces the background material on the existing theory of zigzag categories [20], and presents a new categorical analysis which characterises the universal property of zigzags and abstractly examines contraction. Section 3 introduces the notion of *framed zigzags*, which are a generalisation of the zigzags to an appropriately enriched setting. We define *framed contraction* in Section 3.2, and argue for its correctness by showing that it computes some marked colimit. Section 4 shows that within framed zigzags we can represent invertible generators, and empirically witnesses that the resulting structure is coherent.

1.3 Notation and prerequisites

Familiarity with basic (monoidal) category theory and string diagrams is assumed throughout. Some prior knowledge of enriched category theory and 2-category theory is also helpful, but the technical details can be glossed over without detracting from the main ideas. All (enriched) categories will be strict and small, which means that their objects form a set and can be compared for equality. We typically use font faces like C for categories, and $\underline{C}$ for enriched categories.

2 Zigzag categories

We recall the zigzag construction of Reutter and Vicary [20] and subsequent developments [11, 26], which is the basis for encoding higher-dimensional string diagrams in the proof assistant homotopy.io.

Definition 2.1 (Augmented simplex category). Δ_+ is the category given by (possibly empty) finite linear orders $[n] \coloneqq \{i \in \mathbb{N} \mid i \le n\}$, where $n \in \{-1\} \cup \mathbb{N}$, and monotone maps.

Definition 2.2 (Finite interval category). $\Delta_=$ is the category given by nonempty finite linear orders $[n]$, where $n \in \mathbb{N}$, and top-and-bottom-preserving monotone maps.

Lemma 2.1 (Wraith's duality [28]). *There is an equivalence of categories* $\mathsf{R} \colon \Delta_+ \cong {\Delta_=}^{\mathrm{op}}$ *given by* $[n] \mapsto [n+1]$.

Proof sketch. We can represent the objects $[n]$ of Δ_+ pictorially as iterated cospans of the form:

$$r_0 \longrightarrow s_0 \longleftarrow r_1 \longrightarrow \cdots \longleftarrow r_{n+1},$$

where there are $(n+1)$-many instances of s_i, for $0 \le i < n+1$, and $(n+2)$-many instances of r_j, for $0 \le j \le n+1$. In this pictorial representation, morphisms of Δ_+ are monotone maps between the tips s_i of the cospans; monotonicity means that the induced diagram is planar (i.e. no crossing edges).

Any such monotone map determines (and is determined by) a dual (counterposing in the opposite direction) monotone map between the feet r_j of the cospans, and such a map always preserves top and bottom. □

This pictorial representation of iterated cospans is so important that it deserves its own name: a *zigzag*; its feet are called *regular levels* and its tips are called *singular levels*; the number of cospans forming a zigzag is called its *length*, which is equal to its number of singular levels. We describe the legs of a zigzag as *forward and backward components* f_i and b_i, both of which have codomain given by the singular level s_i, and domains given by regular levels r_i and r_{i+1} respectively. All in all, a zigzag is an iterated cospan whose individual cospans take the form $r_i \xrightarrow{f_i} s_i \xleftarrow{b_i} r_{i+1}$.

Definition 2.3 (Zigzag category). Given a small category C, its *category of zigzags* $\mathrm{Zig}(C)$ has

objects *zigzags* $Z_{[n]}$ of length $n+1$ whose regular and singular levels are annotated with C-objects, and forward and backward components are annotated with compatible C-morphisms, written as an iteration of cospans in C of the form:

$$Z_{[n]}(r_j) \xrightarrow{Z_{[n]}(f_j)} Z_{[n]}(s_j) \xleftarrow{Z_{[n]}(b_j)} Z_{[n]}(r_{j+1}),$$

where $0 \le j < n+1$; 0-length zigzags are given by empty iterations, taking the form $Z_{[-1]}(r_0)$;

morphisms *zigzag maps* $h_\alpha \colon Z_{[m]} \to Z'_{[n]}$ which have two constituents:

- **shape** (*underlying*) monotone maps $\alpha \colon [m] \to [n]$ between singular levels…
- **type** …whose components are annotated with C-morphisms (*singular-singular morphisms* $h_\alpha(s_j) \colon Z_{[m]}(s_j) \to Z'_{[n]}(s_{\alpha(j)})$), along with a C^{op}-annotation for its dual monotone map $\mathrm{R}\alpha \colon \mathrm{R}[m] = [m+1] \leftarrow [n+1] = \mathrm{R}[n]$ between regular levels (*regular-regular morphisms* $h_\alpha(r_{i'}) \colon Z_{[m]}(r_{\mathrm{R}\alpha(i')}) \to Z'_{[n]}(r_{i'})$)[3], such that the resulting planar diagram in C typechecks and commutes; composites of regular-regular and singular-singular morphisms are called *regular-singular morphisms* which are indexed by a pair $(r_i, s_{j'})$;

composition composition of underlying monotone maps along with composition in C.

Write $\mathrm{Zig}^n(C)$ for the n-fold iteration of taking categories of zigzags.

Informally, $\mathrm{Zig}^n(C)$ models the 'space' of combinatorial n-dimensional string diagrams with respect to an algebraic signature encoded by C[4]. Practically, we think of C as a poset of generators presenting an algebraic signature whose ordering is given by boundary inclusion.

An important special case is when $C = \mathbf{1}$ is the terminal category, yielding *unannotated* zigzags.

Lemma 2.2. *We have an isomorphism of categories* $\mathrm{Zig}(\mathbf{1}) \cong \Delta_+$.

Proof. Precisely the pictorial interpretation of Δ_+ as before. ☐

Lemma 2.3 ([20, Lemma 19]). $\mathrm{Zig} \colon \mathbf{Cat} \to \mathbf{Cat}$ *is a functor.*

The action of this functor on morphisms (functors) is *reannotation* of zigzags along the input functor which respects length and underlying monotone maps (shape).

We have described Definition 2.3 briskly because we can recover its precise description by deriving its universal property, determining the functor Zig as a parametric right adjoint[5], which we will now proceed to do. A more explicit description of the zigzag category with helpful diagrams is given by Heidemann, Reutter and Vicary [11, § 2] and Tataru and Vicary [26, § 3].

Definition 2.4 ($\mathbf{Cat}/\Delta_+$). The category of small categories over Δ_+, $\mathbf{Cat}/\Delta_+$, is given on objects by functors $p \colon C \to \Delta_+$[6] and on morphisms by functors $F \colon C \to \mathcal{D}$ making the induced triangle commute.

Such a functor p can be interpreted as a small category C where each object $c \in C$ has an associated simplex $[n]$, and each morphism has an associated monotone map between simplices. Alternatively, we can think of p as a Δ_+*-indexed category*, where the objects and morphisms of C are indexed by simplices and monotone maps: for any $[n]$ there is a 'fibre' consisting of a subset of objects of C living above it, and similarly for monotone maps. Morphisms are then precisely functors $C \to \mathcal{D}$ which are compatible with the Δ_+-indexing on either side, which we call Δ_+*-indexed functors.*

$\mathrm{Zig}(C)$ is a Δ_+-indexed category in a canonical way, along a *shape-projection* functor which discards the type information.

Definition 2.5 (Singular projection). Given a zigzag category $\mathrm{Zig}(C)$, its *singular projection* $\pi_C \colon \mathrm{Zig}(C) \to \Delta_+$ is given by sending each zigzag $Z_{[n]} \mapsto [n]$ and each zigzag map $h_\alpha \mapsto \alpha$.

We will also write Zig for the induced functor $\mathbf{Cat} \to \mathbf{Cat}/\Delta_+$, which maps $C \mapsto \pi_C \colon \mathrm{Zig}(C) \to \Delta_+$, disambiguated by context.

Corollary 2.1. *Every category C admits a unique functor to the terminal category* $! \colon C \to \mathbf{1}$. *The singular projection π_C factors as the composite* $\pi_{\mathbf{1}} \circ \mathrm{Zig}(!)$.

Naturally, one may ask if there is also some sort of *type-projection* functor which discards the shape information of a zigzag category. In fact, this can be defined for any Δ_+-indexed category.

Definition 2.6 (Explosion). Given a Δ_+-indexed category $p \colon C \to \Delta_+$, we define a new category $\mathrm{Expl}(p)$ given by

objects for $c \in C$ such that $p(c) = [n]$,

$$\begin{array}{lll} \text{regular} & c_{[n]}(r_i), & 0 \le i \le n+1, \\ \text{singular} & c_{[n]}(s_j), & 0 \le j < n+1; \end{array}$$

morphisms for $h \colon c \to c' \in C(c,c')$ such that $p(h) = \alpha \colon [m] \to [n]$,

$$\begin{array}{lll} \text{regular-regular} & h_\alpha(r_{i'}) \colon c_{[m]}(r_{\mathrm{R}\alpha(i')}) \to c'_{[n]}(r_{i'}), & 0 \le i' \le \mathrm{R}[n] = n+1, \\ \text{singular-singular} & h_\alpha(s_j) \colon c_{[m]}(s^{[m]}_j) \to c'_{[n]}(s^{[n]}_{\alpha(j)}), & 0 \le j < m+1, \\ \text{regular-singular} & h_\alpha(r_i, s_{j'}) \colon c_{[m]}(r^{[m]}_i) \to c'_{[n]}(s^{[n]}_{j'}), & 0 \le j' < n+1,\ \mathrm{R}\alpha(j') \le i \le \mathrm{R}\alpha(j'+1). \end{array}$$

composition given by composition in C and Δ_+, subject to the requirement that all four triangles in

$$\begin{array}{ccccc} c'_{[n]}(r_{\alpha(j)}) & \xrightarrow{\mathrm{id}_{c'}(r_{\alpha(j)}, s_{\alpha(j)})} & c'_{[n]}(s_{\alpha(j)}) & \xleftarrow{\mathrm{id}_{c'}(r_{\alpha(j)+1}, s_{\alpha(j)})} & c'_{[n]}(r_{\alpha(j)+1}) \\ \uparrow{\scriptstyle h_\alpha(r_{\alpha(j)})} & \nearrow{\scriptstyle h_\alpha(r_j, s_{\alpha(j)})} & \uparrow{\scriptstyle h_\alpha(s_j)} & \nwarrow{\scriptstyle h_\alpha(r_{j+1}, s_{\alpha(j)})} & \uparrow{\scriptstyle h_\alpha(r_{\alpha(j)+1})} \\ c_{[m]}(r_j) & \xrightarrow[\mathrm{id}_c(r_j, s_j)]{} & c_{[m]}(s_j) & \xleftarrow[\mathrm{id}_c(r_{j+1}, s_j)]{} & c_{[m]}(r_{j+1}) \end{array}$$

commute independently whenever they exist. Any composable string of morphisms matches the regular expression[7]

$$(RR)^* | (SS)^* | (RR)^* (RS) (SS)^*,$$

and these relations ensure that the regular-regular/singular-singular/regular-singular (according to the alternation in the

[3]Singular-singular morphisms are indexed by the singular levels of the domain, but regular-regular morphisms are indexed by the regular levels of the **codomain**.

[4]Although, it also contains objects which do not encode any well-formed string diagram.

[5]Whenever C admits a terminal object $\mathbf{1}$, any functor $F \colon C \to \mathcal{D}$ canonically factors as a composite $C \xrightarrow{F_1} C/F\mathbf{1} \xrightarrow{\mathrm{dom}} \mathcal{D}$. We call F a *parametric right adjoint* if F_1 is a right adjoint. Parametric right adjoints preserve connected limits.

[6]For ease of typesetting, sometimes we write C where we formally mean p, e.g. when describing a Hom object of $\mathbf{Cat}/\Delta_+$.

[7]Because morphisms monotonically proceed from regular to singular.

match of the regular expression) morphism with the appropriate subindices in the result of composition is uniquely determined.

Each object $c \in C$, where $p(c) = [n]$, has been 'exploded' into a family of $2(n+1)+1$ objects, and similarly for morphisms. When $C = \mathrm{Zig}(\mathcal{D})$, with p its corresponding singular projection, then the objects and morphisms are precisely those in $\mathcal{D}$ annotated by their position within some zigzag/zigzag map — this connection will be explicated in Lemma 2.4.

Expl extends to a functor $\mathbf{Cat}/\Delta_+ \to \mathbf{Cat}$, where Δ_+-indexed functors $C \to \mathcal{D}$ are sent to 'reannotation' functors $\mathrm{Expl}(C) \to \mathrm{Expl}(\mathcal{D})$.

Lemma 2.4 (Universal property of zigzag categories). *We have an adjunction as follows:*

$$\mathbf{Cat} \underset{\mathrm{Zig}}{\overset{\mathrm{Expl}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Cat} \,/\, \mathbf{\Delta}_+ \,.$$

We obtain the endofunctor Zig on **Cat** as the parametric right adjoint given by composition with the domain projection:

$$\mathrm{Zig} \colon \mathbf{Cat} \underset{\mathrm{Zig}}{\overset{\mathrm{Expl}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Cat} \,/\, \mathbf{\Delta}_+ \xrightarrow{\mathrm{dom}} \mathbf{Cat} \,.$$

Our notation has been chosen in order to make this proof simple, but in essence it formalises the idea that the data in a zigzag category $\mathrm{Zig}(C)$ is nothing more than packaging the data of C in a zigzag shape determined by the structure of Δ_+. For instance, a choice of $Z_{[n]} \in \mathrm{Zig}(C)$, with length $n+1$, is equivalent to a global element $Z \colon \mathbf{1} \to \mathrm{Zig}(C)$; this bijection says that this data (i.e. $\mathbf{J} = \mathbf{1}$) is precisely an $(n+1)$-length iterated cospan in C, because in this case $\mathrm{Expl}(\pi_C \circ Z)$ is the category which embodies the shape of an $(n+1)$-length iterated cospan. Zigzag maps can be recovered from this adjunction by setting $\mathbf{J} = \cdot \to \cdot$ to be the walking arrow category (a.k.a. the simplex $[1]$ seen as a category), and similarly so can composition ($\mathbf{J} = [2]$) and associativity of composition ($\mathbf{J} = [3]$), and so on. We use these correspondences implicitly to consider the *explosion* of a zigzag (map).

Note that the bijection of Lemma 2.4 can be iterated, as we often do when considering n-fold zigzag categories $\mathrm{Zig}^n(C)$:

$$\frac{F \colon \mathbf{J} \to \mathrm{Zig}^n(C)}{\int^k F \colon \mathrm{Expl}^k(\pi_C \circ F \colon \mathbf{J} \to \Delta_+) \to \mathrm{Zig}^{n-k}(C)}, \quad 0 \le k \le n.$$

We call this the *k-fold exploded diagram* of F, or the *full explosion* when $n = k$.

Finally, a n-fold zigzag $Z_{[m]} \in \mathrm{Zig}^n(C)$ is *globular* (with respect to C) if every regular-regular morphism in its k-fold explosion, for each $k \le n$, is an identity (which is vacuously true when $n < 2$). The original definition of zigzags [20] required this globularity to hold by imposing this constraint directly in the definition of zigzag map, but it is inessential in our development of the theory, although any n-dimensional string diagram (encoded as an n-fold zigzag) that we consider is necessarily globular.

2.1 Contraction

Given some globular zigzag $Z_{[n]} \in \mathrm{Zig}(C)$, where $n \ge 0$, we wish to determine whether there is some *universal* 1-length globular zigzag and corresponding zigzag map $c \colon Z_{[n]} \to Z'_{[0]}$. Operationally, this corresponds to canonically shrinking a string diagram to a smaller one along a 'homotopy' — hence 'contraction'. The corresponding exploded diagram is as in Figure 2. All these data are derived by

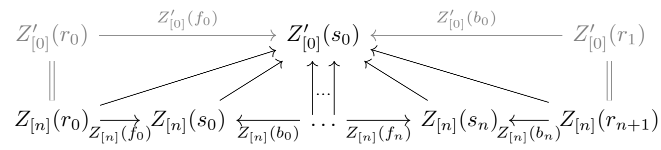


Figure 2: Contraction as universal zigzag shortening via colimits.

asking for $Z'_{[0]}(s_0)$ to be the colimit of the exploded zigzag $Z_{[n]}$ in C, with the extra pieces arising from globularity (as we have alluded to with our shading). If this colimit does not exist, then the contraction operation is undefined.

In essence, colimits in $\mathrm{Zig}(C)$ should be constructed from colimits in C by the following two-step procedure: given some diagram $F \colon \mathbf{J} \to \mathrm{Zig}(C)$,

(1) first, compute the underlying shape of the desired colimit by computing the colimit of $\mathbf{J} \xrightarrow{F} \mathrm{Zig}(C) \xrightarrow{\pi_C} \Delta_+$;
(2) use colimits derived from $\int F \colon \mathrm{Expl}(\pi_C \circ F) \to C$ to fill in this shape with appropriate data (type) from C to make this a universal cocone in $\mathrm{Zig}(C)$.

At each step, the procedure may fail, indicating the non-existence of the colimit in question. Through iterated explosion, this gives a procedure to recursively determine colimits in $\mathrm{Zig}^n(C)$ in terms of colimits in C.

Reutter and Vicary [20] present a series of technical lemmata which describe this procedure for globular zigzags, manipulating zigzags directly, and show that it is sound and complete (a colimit exists and is found by the procedure exactly when it succeeds, and otherwise the procedure fails). Instead, we will use some 2-category theory to slickly reconstruct their results. We note that our reconstruction does not require globularity, and the shaded parts of Figure 2 arise from the left Kan extension we build.

First, we establish some 2-categorical preliminaries.

Definition 2.7. There is a strict 2-category **Cat** of small categories, functors, and natural transformations. Explicitly, its Hom categories are functor categories $[C, \mathcal{D}]$ of functors and natural transformations.

Lemma 2.5. *Strict 2-functors preserve adjunctions.*

Definition 2.8 (Slice Hom category). Given two functors $p \colon \mathcal{A} \to C$ and $q \colon \mathcal{B} \to C$, we define the (strict) slice Hom category $[p,q]_{/C}$ to have objects functors $I \colon \mathcal{A} \to \mathcal{B}$ satisfying $q \circ I = p$ and morphisms given by natural transformations.

This is a special case of the strict slice 2-category, over the strict 2-category **Cat**. Because of this strictness, a morphism is *any* natural transformation of appropriate type (without further conditions). Moreover, this determines a strict 2-category $\mathbf{Cat}/C$ for any small category C, along with a forgetful strict 2-functor $\mathrm{dom} \colon \mathbf{Cat}/C \to \mathbf{Cat}$ sending $p \colon \mathcal{A} \to C \mapsto \mathcal{A}$.

Lemma 2.6. $\mathrm{dom} \colon \mathbf{Cat}/C \to \mathbf{Cat}$ *is locally fully faithful, i.e, every induced functor between Hom categories* $[p,q]_{/C} \to [\mathrm{dom}(p), \mathrm{dom}(q)]$ *is fully faithful.*

Lemma 2.7. *Locally fully faithful strict 2-functors reflect adjunctions: if $F : \mathcal{C} \to \mathcal{D}$ is such, and for some morphisms f and g in $\mathcal{C}$ $Ff \dashv Fg$ in $\mathcal{D}$, then $f \dashv g$ in $\mathcal{C}$.*

Corollary 2.2. *An adjunction in $\mathbf{Cat}/C$ between $p : \mathcal{A} \to C$ and $q : \mathcal{B} \to C$ is precisely a pair of functors $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ satisfying $q \circ F = p$ and $p \circ G = q$ (they are morphisms in $[p,q]_{/C}$ and $[q,p]_{/C}$ respectively), and $F \dashv G$ in $\mathbf{Cat}$ (they are adjoint functors).*

Proposition 2.1. Zig *extends to a strict 2-functor.*

2.1.1 Contraction recursive case — shape.

Proposition 2.2. *A category C admits a terminal object if and only if the terminal functor $! : C \to \mathbf{1}$ admits a right adjoint.*

Corollary 2.3. *If C admits a terminal object, then the singular projection $\pi_C : \mathrm{Zig}(C) \to \Delta_+$ preserves colimits.*

In other words, if C admits a terminal object[8], then the existence of a colimit of $F : \mathbf{J} \to \mathrm{Zig}(C)$ can be refuted by first projecting into Δ_+ along the singular projection: the colimit of F can only exist if the composite admits a colimit in Δ_+. Such colimits are determined by computation in **Pos** (a cocomplete category) along the embedding $\Delta_+ \hookrightarrow \mathbf{Pos}$ and then checking that the result is some n-simplex rather than an arbitrary poset [24].

We also want this to be uniform, for considering iterated zigzag categories.

Lemma 2.8. *If C admits a terminal object, then so does $\mathrm{Zig}(C)$.*

Inductively, this will hold for all $\mathrm{Zig}^n(C)$.

This justifies step 1 of the recursive contraction procedure outlined in Section 2.1.

2.1.2 Contraction recursive case — type.

Now we proceed to justify step 2. Firstly, we recast colimits as certain kinds of left Kan extension. Fix a small category $\mathbf{J}$.

Definition 2.9 (Under cocone category). The *under cocone category* $\mathbf{J}^{\triangleright}$ is given by freely adjoining a terminal object 1 to $\mathbf{J}$, and is equipped with an inclusion $i : \mathbf{J} \hookrightarrow \mathbf{J}^{\triangleright}$.

Proposition 2.3. *For any functor $F : \mathbf{J} \to C$, the data of a colimit of F (i.e. a universal cocone $\iota : F \Rightarrow \mathrm{const}_{\mathrm{colim}F}$ under F) is equivalent to the left Kan extension $\mathrm{Lan}_i F$[9]:*

$$\begin{array}{ccc} \mathbf{J} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle i}\downarrow & \nearrow_{\mathrm{Lan}_i F} & \\ \mathbf{J}^{\triangleright} & & \end{array} \qquad \text{given by} \quad \mathrm{Lan}_i F(j) = \begin{cases} Fj, & j \in \mathbf{J}, \\ \mathrm{colim}F, & j = 1, \end{cases}$$

and $\mathrm{Lan}_i F(f) = Ff$ for morphisms of $\mathbf{J}$, $\mathrm{Lan}_i F(! : j \to 1) = \iota_j$ for each unique morphism in $\mathbf{J}^{\triangleright}$.

Now we justify abstractly that lifting colimits through explosion is well-defined and meaningful.

Theorem 2.10. *Let $F : \mathbf{Cat} \to \mathbf{Cat}$ be a parametric right adjoint endofunctor, which factors as:*

$$F : \mathbf{Cat} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Cat}/F1 \xrightarrow{\mathrm{dom}} \mathbf{Cat},$$

with the natural bijection of the adjunction denoted:

$$\forall C \in \mathbf{Cat}, p : \mathcal{E} \to F\mathbf{1} \in \mathbf{Cat}/F\mathbf{1}. \quad [L(p), C] \overset{\theta}{\cong} [p, R(C)]_{/F\mathbf{1}}.$$

For a $F\mathbf{1}$-indexed categories $p : \mathcal{E} \to F\mathbf{1}$ and $q : \mathcal{E}' \to F\mathbf{1}$, and $F\mathbf{1}$-indexed functors $G : p \to R(C)$ and $i : p \hookrightarrow q$, where i is an inclusion[10]*, we have a bijection of commuting diagrams in* $\mathbf{Cat}$*:*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\mathrm{dom}(G)} & \mathrm{dom}(R(\mathcal{C})) \\ {\scriptstyle \mathrm{dom}(i)}\downarrow & \nearrow_{\mathrm{Lan}_{\mathrm{dom}(i)} \mathrm{dom}(G)} & \\ \mathcal{E}' & & \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} L(p) & \xrightarrow{\theta^{-1}(G)} & \mathcal{C} \\ {\scriptstyle L(i)}\downarrow & \nearrow_{\mathrm{Lan}_{L(i)} \theta^{-1}(G)} & \\ L(q), & & \end{array}$$

i.e. $\mathrm{Lan}_{\mathrm{dom}(i)} \mathrm{dom}(G) \cong \mathrm{dom}(\theta(\mathrm{Lan}_{L(i)} \theta^{-1}(G)))$.

Corollary 2.4 (Colimits in $\mathrm{Zig}(C)$ from colimits in C). *Left Kan extensions along inclusions i (which subsume colimits) in the Δ_+-indexed category $\mathrm{Zig}(C)$ are computed as left Kan extensions along $\mathrm{Expl}(i)$ in C: for $F : \mathbf{J} \to \mathrm{Zig}(C)$ and $i : \pi_C \circ F \hookrightarrow \pi_C \circ \mathrm{Lan}_{\mathrm{dom}(i)} F$ ($\mathrm{dom}(i) : \mathbf{J} \hookrightarrow \mathbf{J}'$),*

$$\frac{\mathrm{Lan}_{\mathrm{dom}(i)} F : \mathbf{J}' \to \mathrm{Zig}(C)}{\overline{\mathrm{Lan}_{\mathrm{Expl}(i)}(\int F) : \mathrm{Expl}(\pi_C \circ \mathrm{Lan}_{\mathrm{dom}(i)} F) \to C}}$$

In the case of colimits in $\mathrm{Zig}(C)$, this is tantamount to a collection of colimits in C.

Again, this will work inductively for all $\mathrm{Zig}^n(C)$.

This justifies step 2 of the contraction procedure outlined in Section 2.1, and completes our description of the relationship between colimits and exploded zigzags. Together, these determine the *recursive* case of the contraction procedure, recursive on the dimension n for a diagram in $\mathrm{Zig}^n(C)$.

2.1.3 Contraction base case — accidental colimits and corestriction.

Eventually, these colimits in $\mathrm{Zig}^n(C)$ boil down to a collection of colimits in C (a poset of generators and boundary inclusions encoding an algebraic signature), which is the *base* case of the procedure.

Contraction is supposed to be a universal zigzag shrinking operation, so it should not 'create new algebraic information'. For example, let C be the poset $x \to f \leftarrow y \to g \leftarrow z$, which encodes the signature consisting of two composable 1-cell generators $f : x \to y$ and $g : y \to z$. In $\mathrm{Zig}(C)$, we can form the 1-dimensional string diagram which represents the composite as the zigzag of length 2 above. Contracting this zigzag would be by computation of a colimit in the poset C. There is no such colimit (a join of f and g), and indeed we would not expect a 1-length zigzag (a contraction) which universally represents $g \circ f$.

However, if C contains an additional $\alpha \geq f, g$, modelling a 2-cell generator $\alpha : f \Rightarrow g$, then it may happen that this colimit exists *accidentally*[11]. Of course, it is still nonsense for $x \to f \leftarrow y \to g \leftarrow z$ to contract, so in order to explain this phenomenon we need to refine exactly where the colimit computed by the base case of contraction takes place.

Supposing that $F : \mathbf{J} \to C$ is this diagram in C, with the observation that the 'no new algebraic information' requirement is tantamount to building a colimit solely from the existing pieces of this diagram,

[8] By design, C is chosen to be a poset in practice for which this condition is easy to check.
[9] Because i is an inclusion, the unit 2-morphism associated to the left Kan extension can be chosen to be the identity, and this triangle commutes strictly.
[10] i.e. $\mathrm{dom}(i) : \mathcal{E} \hookrightarrow \mathcal{E}'$ is a subcategory inclusion.
[11] Worse still, this colimit stops existing if there is another 2-cell generator $\beta : f \Rightarrow g$; contraction of these diagrams should not depend on the existence of these higher cells which are not mentioned by the diagram!

it transpires that we actually want a colimit of the *corestriction* of F to its image: F factors as

$$\mathbf{J} \xrightarrow{F} \mathcal{C}, \qquad \mathbf{J} \xrightarrow{F|^{\mathrm{im}(F)}} \mathrm{im}(F) \hookrightarrow \mathcal{C}.$$

Such a colimit of $F|^{\mathrm{im}(F)}$ will satisfy the property that it only uses data in its image.

This corestriction is compatible with the zigzag construction because we can corestrict through C via explosion as follows:

$$\dfrac{\dfrac{F\colon D(C(J)) \to \mathrm{Zig}(C)}{\int F\colon \mathrm{Expl}(\underline{\pi}_C \circ F) \to C}\ \mathrm{Expl} \dashv \mathrm{Zig}}{\dfrac{\int F|^{\mathrm{im}(\int F)}\colon \mathrm{Expl}(\underline{\pi}_C \circ F) \to \mathrm{im}(\int F)}{F|^{\mathrm{im}(\int F)}\colon D(C(J)) \to \mathrm{Zig}(\mathrm{im}(\int F))}\ \mathrm{Expl} \dashv \mathrm{Zig}}\ \text{corestrict}$$

It can also be iterated for $\mathrm{Zig}^n(C)$.

Operationally, this has a simple meaning: the algebraic signature C at the base of the zigzag $\mathrm{Zig}^n(C)$ in which we are performing contraction is not one of its inputs; rather, contraction derives (uniquely) the minimal fragment of the signature within which its input can be expressed, and the result is only a colimit with respect to that setting.

Without loss of generality, we therefore assume that, for contraction (and later, framed contraction), every diagram $F\colon \mathbf{J} \to \mathrm{Zig}^n(C)$ has a corresponding full explosion in C which is surjective.

The remainder of this paper is about deriving an analogous contraction operation for *framed* zigzags, which we will define in Section 3.

3 Enriched categories of framed zigzags

In Section 2, C at the base of the iterated zigzag construction consists of generators and boundary inclusions, encoding an algebraic signature for which we can 'type' combinatorial encodings of n-dimensional string diagrams in $\mathrm{Zig}^n(C)$. For all intents and purposes, in the original construction it is merely a poset. When we wish to encode inverses of generators in this poset, an issue arises — posets are *thin*: either $p \le q$ or $p \nleq q$; that is, the Homs are boolean-valued, and this loses the information of *how* a particular generator is on the boundary of another (only recording that it either is or is not). The obvious solution is to make C a proper category, but this breaks globular zigzags in $\mathrm{Zig}^n(C)$ for $n \ge 2$ in a fundamental way: globularity requires that regular-regular maps are annotated by identity morphisms in C, but the definition of zigzag map requires *commutativity* in C which essentially forces every regular-singular map to be equal. Taking inspiration from 2-category theory, we would instead seek to make C a 2-poset (a.k.a. a **Pos**-category): a categorified version of a poset whereby there Homs are not boolean but themselves posets[12].

3.1 Pos-enriched and $\bigvee$Lat-enriched framed zigzags

The setting to study this is **Pos**-enriched category theory, and this enables us to relax the commutativity condition in the definition of zigzag map, as **Pos**-categories admit a notion *(op)laxly* commuting diagram. We write $f \Rightarrow g$ for the ordering relation in Hom posets, and call this witness a *2-cell filler* in analogy to 2-category theory.

Unfortunately, a naive '**Pos**-enriched' version of $\mathrm{Zig}(C)$ is not well-defined as composition will not exist in general; however, the sketch is still useful for intuition, and we will fix this issue later.

*Sketch 3.1 (Naive '**Pos**-enriched' framed zigzags).* Given a small **Pos**-category $\underline{C}$, attempt to define a **Pos**-category by

objects identical to zigzags as in Definition 2.3 ;

morphisms analogous to zigzag maps as in Definition 2.3, but requiring only *oplax* commutativity in $\underline{C}$: in the exploded diagram in $\underline{C}$, between a regular-regular level and a singular-singular level there may be many paths; we require that 2-cell fillers exist in $\underline{C}$ between the $\underline{C}$-morphisms annotating these paths along the converse of the path-short-circuiting relation, i.e. 2-cell fillers point from shorter paths to longer paths[13]. Identity morphisms are given by identity morphisms as regular-regular and singular-singular components; regular-singular components are given by composition, making all 2-cell fillers trivial.

2-cell fillers $\underline{h_\alpha} \Rightarrow \underline{h'_\beta}$ exactly when the underlying monotone maps α and β are equal and every $\underline{C}$-component of $\underline{h_\alpha}$ is pointwise below the corresponding $\underline{C}$-component of $\underline{h'_\beta}$.

composition fails to exist in general, because due to oplaxity there may be many incomparable maps between any pair of regular and singular levels, and there is no way to make a canonical choice for the result regular-singular $\underline{C}$-annotation; e.g. the shape of the composition in Figure 3 is determined, but when trying to determine the $\underline{C}$-annotation in the result: in the input to composition, there are many distinct $\underline{C}$-morphisms $r_0 \to s''_0$, but there is no canonical way of choosing or combining them.

In practice, every composite we require for framed contraction will exist, but rather than attempting to make composition partially defined, we instead 'complete' our setting to add all the missing composites formally. We do this by first freely moving from **Pos**-enrichment to $\bigvee$**Lat**-enrichment.

$\bigvee$**Lat** is the category of $\vee$-semilattices which admit **nonempty** joins (i.e. the binary operation $\vee$ is associative but not necessarily unital) and $\vee$-preserving maps. Let $\downarrow\colon$ **Pos** $\to \bigvee$**Lat** be the functor which takes a poset to its $\vee$-semilattice of nonempty downwards-closed subsets ordered by inclusion, with $\vee$ given by set union. For a poset P, we denote the *principal downset* of $p \in P$ as $\downarrow p := \{q \in P : q \le p\} \subseteq P$. $\downarrow$ is left-adjoint to the forgetful functor $U^\vee\colon \bigvee\mathbf{Lat} \to \mathbf{Pos}$.

Lemma 3.1. *The following are monoidal adjunctions:*

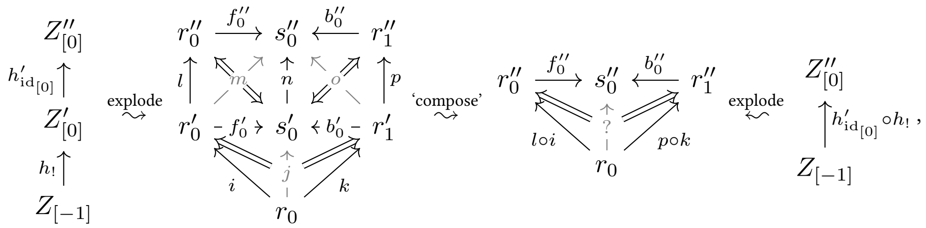
$$(\mathbf{Set}, \times, 1) \underset{U^{\le}}{\overset{\Pi_0}{\underset{\perp}{\overset{\perp}{\rightleftarrows}}}} (\mathbf{Pos}, \times, 1) \underset{U^{\vee}}{\overset{\downarrow}{\underset{\perp}{\rightleftarrows}}} (\textstyle\bigvee \mathbf{Lat}, \otimes, 1),$$

where $\otimes$ is the free semilattice tensor product (Lemma F.2), D is the discrete poset functor, $U^{\le}$ is the underlying set functor, and Π_0 is the connected components functor.

We won't use the monoidal structure of these categories aside from the observation that these adjunctions are monoidal and hence

[12] This is akin to the difference between the face poset and the 2-poset of entrance paths (Appendix A), with respect to a CW decomposition of a space.

[13] Our directional convention for these 2-cell fillers is arbitrary. Our choice aligns with greater commonality of oplax colimits in the literature.



where $i=h_!(r_0)$, $j=h_!(r_0,s_0)$, $k=h_!(r_1)$, $l=h'_{\mathrm{id}_{[0]}}(r_0)$, $m=h'_{\mathrm{id}_{[0]}}(r_0,s_0)$, $n=h'_{\mathrm{id}_{[0]}}(s_0)$, $o=h'_{\mathrm{id}_{[0]}}(r_1,s_0)$, $p=h'_{\mathrm{id}_{[0]}}(r_1)$.

Figure 3: Failure of composition for Sketch 3.1.

induce change-of-base adjunctions between enriched categories (Lemma F.3), and elect to stop writing monoidal structure explicitly.

Corollary 3.1. *We have adjunctions of 2-categories:*

$$\mathbf{Cat} \underset{U_*^{\leq}}{\overset{\Pi_{0*}}{\underset{\perp}{\overset{\perp}{\leftrightarrows}}}} D_* \; \mathbf{PosCat} \underset{U_*^{\vee}}{\overset{\downarrow_*}{\underset{\perp}{\rightleftarrows}}} \bigvee\mathbf{LatCat},$$

where each functor is the identity on objects.

This tells us that from any **Pos**-category, we can freely obtain a $\bigvee$**Lat**-enriched category with the same objects: one where the Hom posets $\underline{C}(X,Y)$ are replaced with Hom $\vee$-semilattices whose elements are nonempty downwards-closed subsets $S\subseteq\underline{C}(X,Y)$ with joins given by unions, ordered by inclusion. Moreover, we could also alternatively start from a category, which trivialises the downwards-closed condition ($\downarrow\circ D$ is the nonempty powerset functor). We use this to determine what shapes of framed zigzags should be, i.e. the appropriate enriched version of Δ_+.

Definition 3.2. $\underline{\Delta_+}$ is the $\bigvee$**Lat**-category given by $\downarrow_*(D_*(\Delta_+))$. Explicitly, it has (possibly empty) simplices $[n]$ as objects; non-empty sets of monotone maps $\{\alpha\colon [m]\to[n]\}$ with composition given by the set of all pairwise composites and identity given by singleton sets containing identity monotone maps; joins given by set unions; and ordering[14] given by set inclusion.

Now we are ready to define a genuine $\bigvee$**Lat**-category of framed zigzags, fixing Sketch 3.1.

*Definition 3.3 (Framed zigzag $\bigvee$**Lat**-category).* Given a small $\bigvee$**Lat**-category $\underline{C}$, the $\bigvee$**Lat**-category of framed zigzags $\underline{\mathrm{Zig}(C)}$ has

objects *framed zigzags* $Z_{[n]}$ — the same as in Sketch 3.1;

morphisms *framed zigzag maps* $h_A\colon Z_{[m]}\to Z'_{[n]}$ — a *shape*, given by a nonempty set of monotone maps $A\subseteq\Delta_+([m],[n])$ (a morphism of $\underline{\Delta_+}$), and a *type* associating to each element of $\alpha\in A$ a $\underline{C}$-annotation in the vein of Sketch 3.1;

join $\bigvee_{i\in I\neq\emptyset}h_{iA_i}$ is a framed zigzag map whose shape is $\bigcup_{i\in I}A_i$ (take the join in $\underline{\Delta_+}$), and type associates to each $\alpha\in\bigcup_{i\in I}A_i$ the $\underline{C}$-annotation obtained by taking the pointwise join in $\underline{C}$ of the $\underline{C}$-annotations of $\{h_{j_{A_j}}\}_{j\in J}$, where $J:=\{j\in I:\alpha\in A_j\}$;

composition determine the shape A by composition in $\underline{\Delta_+}$, analogously to Definition 2.3, and then for each component $\underline{C}$-annotation associated each to $\alpha\in A$, determining its type by taking the join of all possible composites; e.g. for Figure 3, set $?:=(f''_0\circ l\vee n\circ f'_0)\circ i\vee(b''_0\circ p\vee n\circ b'_0)\circ k$[15].

This can be verified to satisfy the axioms of a $\bigvee$**Lat**-category, using the fact that composition in $\bigvee$**Lat**-categories is $\vee$-preserving.

Analogously to the unenriched case, $\underline{\mathrm{Zig}(C)}$ is naturally equipped with a $\bigvee$**Lat**-enriched singular projection $\underline{\pi_C}\colon\underline{\mathrm{Zig}(C)}\to\underline{\Delta_+}$, and this construction determines a functor $\underline{\mathrm{Zig}}\colon\bigvee\mathbf{LatCat}\to\bigvee\mathbf{LatCat}/\underline{\Delta_+}$. We also have a $\bigvee$**Lat**-enriched version of Definition 2.6.

Definition 3.4 (Framed explosion). Given a $\underline{\Delta_+}$-indexed $\bigvee$**Lat**-category $\underline{p}\colon\underline{C}\to\underline{\Delta_+}$, define a new $\bigvee$**Lat**-category $\underline{\mathrm{Expl}}(\underline{p})$ given by

objects analogous to Definition 2.6;

morphisms morphisms come in three varieties as per Definition 2.6, but are indexed by nonempty subsets of monotone maps $A\subseteq\Delta_+([m],[n])$ instead of individual monotone maps $\alpha\in\Delta_+([m],[n])$, with the components generated existentially; e.g. for singular-singular components, morphisms are generated for each $\alpha\in A$ of the form:
$$h_A(s_j)\colon c_{[m]}(s_j^{[m]})\to c'_{[n]}(s_{\alpha(j)}^{[n]})\quad\text{such that }0\leq j<m+1;$$

join set union on the indexing sets of monotone maps;

composition analogous to Definition 2.6, except replacing commuting triangles with oplaxly commuting triangles.

Like before, this determines a left adjoint to $\underline{\mathrm{Zig}}$:
$$\underline{\mathrm{Expl}}\colon\bigvee\mathbf{LatCat}/\underline{\Delta_+}\to\bigvee\mathbf{LatCat}.$$
Using sliced adjunctions (Lemma F.4), the whole situation is rendered nicely as Figure 4: the unenriched Expl ⊣ Zig is obtained from $\underline{\mathrm{Expl}}\dashv\underline{\mathrm{Zig}}$ by composition of these adjunctions, and moreover we obtain a **Pos**-enriched version[16], and each version is a parametric right adjoint.

The free $\vee$-completion of Hom posets under $\downarrow_*$ is just a technical device to ensure that composition in Definition 3.3 is well-defined, and it bears no relevance to the framed contraction we will introduce, so we focus on the **Pos**-enriched version of the framed zigzag construction from hereon.

[14] This is already determined by defining the join, as in any $\vee$-semilattice $x\leq y\iff x\vee y=y$. As such, when we define a $\bigvee$**Lat**-category, it suffices to define only join and elide ordering.

[15] This expression has been simplified to include only maximal incomparables.

[16] Explicitly, this begins with a **Pos**-cateogry, which is freely completed via $\downarrow_*$ to a $\bigvee$**Lat**-category, from which we take the framed zigzag $\bigvee$**Lat**-category (Definition 3.3), which is then forgetfully seen as a **Pos**-category living over $D_*(\Delta_+)$. The sliced right adjoint $U_*^{\vee}/_{D_*(\Delta_+)}$ takes a $\underline{\Delta_+}$-indexed $\bigvee$**Lat**-category and uses pullbacks in **PosCat** to 'delete' the parts which do not live above singleton sets of monotone maps.

$$\begin{array}{ccccccccccc}
\mathbf{Cat} & \underset{D_*}{\overset{\Pi_{0*}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} & \mathbf{PosCat} & \underset{\downarrow_*}{\overset{U^\vee_*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} & \bigvee\mathbf{LatCat} & \underset{\underline{\mathrm{Zig}}}{\overset{\underline{\mathrm{Expl}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} & \bigvee\mathbf{LatCat}\,/\,\underline{\mathbf{\Delta}_+} & \underset{U^\vee_*/D_*(\mathbf{\Delta}_+)}{\overset{\downarrow_*/D_*(\mathbf{\Delta}_+)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} & \mathbf{PosCat}\,/D_*(\mathbf{\Delta}_+) & \underset{U^{\leq}_*/\mathbf{\Delta}_+}{\overset{D_*/\mathbf{\Delta}_+}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} & \mathbf{Cat}\,/\,\mathbf{\Delta}_+ \xrightarrow{\mathrm{dom}} \mathbf{Cat} \\
 & & & & & & \downarrow \mathrm{dom} & & \downarrow \mathrm{dom} & & \\
 & & & & & & \bigvee\mathbf{LatCat} & & \mathbf{PosCat} & &
\end{array}$$

Figure 4: Relationship between unenriched and enriched versions of the zigzag construction.

3.1.1 Unframing. We can obtain a left adjoint to $D_*/_{\Delta_+}$ also.

LEMMA 3.2. *Define* $\Pi_{0*}/_{D_*(\Delta_+)} : \mathbf{PosCat}/D_*(\Delta_+) \to \mathbf{Cat}/\Delta_+$ *by applying* Π_{0*}*; then*

$$\begin{array}{ccc}
\mathbf{Cat}\,/\,\mathbf{\Delta}_+ & \underset{D_*/\mathbf{\Delta}_+}{\overset{\Pi_{0*}/D_*(\mathbf{\Delta}_+)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} & \mathbf{PosCat}\,/D_*(\mathbf{\Delta}_+) \\
\mathrm{dom}\downarrow & & \downarrow\mathrm{dom} \\
\mathbf{Cat} & \underset{D_*}{\overset{\Pi_{0*}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} & \mathbf{PosCat}
\end{array}$$

commutes.

Using the pointwise nature of $\underline{\mathrm{Zig}}$, we can relate the framed and unframed versions of the zigzag construction as follows.

THEOREM 3.5 (UNFRAMING). $\Pi_{0*}/_{D_*(\Delta_+)} \circ \underline{\mathrm{Zig}} = \mathrm{Zig} \circ \Pi_{0*}$.

We call this *unframing*, and it is a kind of conservativity result for framed zigzags over zigzags.

3.2 Framed contraction

As in Section 2.1, the enriched analogue of contraction should construct 'colimits' in $\underline{\mathrm{Zig}}(\underline{C})$ from 'colimits' in $\underline{C}$. However, as is the typical case in enriched category theory, the idea of a colimit as a universal cocone requires further nuance, especially given that under **Pos**-enrichment we have oplaxly commuting triangles in our (**Pos**-enriched) diagrams.

Here, we make precise the notion of *oplaxly commuting diagram*, using the underlying graph of categories and the usual notion of *commuting diagram*.

Definition 3.6 (Free category and underlying graph). There is a monadic adjunction $U^\circ : \mathbf{Cat} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Graph} : C$ which takes a category to its underlying (directed) graph, and a (directed) graph to the free category of paths and path concatenation generated by it.

Definition 3.7 (Commuting diagram). A *commuting diagram* in a category C is a functor which factors via a poset (thin category): $C(J) \to P \to C$ for some simple[17] directed **acyclic** graph J and poset P.

Acyclicity encodes the idea that posets are antisymmetric, so there are no non-trivial isomorphisms in P.

Definition 3.8 (Oplax 2-poset, atoms). An oplax 2-poset is a **Pos**-category $\underline{P}$ with the same structure as $D(C(J))$ for some directed acyclic graph J, except additionally for each pair of paths $p,p' \in \underline{P}(u,v)$, $p \Rightarrow p'$ whenever p can be obtained from p' by deleting vertices. We describe the minimal elements of $\underline{P}(u,v)$ as *atoms*, and the composite of any pair of non-identity morphisms is never an atom.

[17]The edge relation is boolean.

Definition 3.9 (Oplaxly commuting diagram). An oplaxly commuting diagram in a **Pos**-category $\underline{C}$ is a **Pos**-functor which factors via an oplax 2-poset: $D(C(J)) \to \underline{P} \to \underline{C}$ for some simple directed acyclic graph J and oplax 2-poset $\underline{P}$.

When convenient, we will assume that the first composite of an oplaxly commuting diagram is the identity without loss of generality, i.e. taking the definition of oplaxly commuting diagram to be **Pos**-functors $\underline{P} \to \underline{C}$ for some oplax 2-poset $\underline{P}$.

We carve out the appropriate notion of colimit in this section, along with an effective procedure for constructing them.

3.2.1 Oplax conical colimits. Fix a pair of **Pos**-functors $F,G : \mathbf{J} \rightrightarrows \underline{C}$.

Definition 3.10 (Lax natural transformation). A *lax natural transformation* $\alpha : F \Rightarrow G$ is a family of $\underline{C}$-morphisms $\alpha_j : Fj \to Gj$ for each $j \in \mathbf{J}$, equipped with lax naturality squares: for each morphism $f : j \to j'$ in $\mathbf{J}$, a 2-cell filler $Gf \circ \alpha_j \Rightarrow \alpha_{j'} \circ Ff$ in $\underline{C}$:

$$\begin{array}{ccc}
Fj & \xrightarrow{Ff} & Fj' \\
\alpha_j\downarrow & \Longrightarrow & \downarrow\alpha_{j'} \\
Gj & \xrightarrow[Gf]{} & Gj'.
\end{array}$$

The collection of all lax natural transformations $F \Rightarrow G$ form a poset $\mathbf{Lax}(F,G)$ ordered component-wise. Note that the lax naturality squares at identity morphisms are necessarily strictly commuting.

Definition 3.11 (Oplax cocone). An *oplax cocone* over $c \in \underline{C}$ is a lax natural transformation $\alpha : F \Rightarrow \mathrm{const}_c$. That is, a family of $\underline{C}$-morphisms $\alpha_j : Fj \to c$ such that for each morphism $f : j \to j'$ in $\mathbf{J}$, there is a 2-cell filler $\alpha_j \Rightarrow \alpha_{j'} \circ Ff$:

$$\begin{array}{ccc}
Fj & \xrightarrow{Ff} & Fj' \\
 & \Rightarrow & \\
\alpha_j \searrow & & \swarrow \alpha_{j'} \\
 & c. &
\end{array}$$

Oplax cocones over a diagram F with a fixed tip c are partially-ordered pointwise, and form a poset $\mathbf{OplaxCocone}(F,c)$.

Definition 3.12 (Oplax conical colimit). The *oplax conical*[18] *colimit* of F is the representing object (unique up to unique isomorphism) of the **Pos**-functor

$$\mathbf{OplaxCocone}(F,-) : \underline{C} \to \mathbf{Pos},$$

i.e. an object oplax colim$F \in \underline{C}$ such that there is an order isomorphism of posets $\underline{C}(\text{oplax colim}\,F, c) \cong \mathbf{OplaxCocone}(F,c)$ natural in each $c \in \underline{C}$.

Unpacking, this is equivalent to a *universal* oplax cocone $\iota : F \Rightarrow \mathrm{const}_{\text{oplax colim}F}$: for any other oplax cocone $\alpha : F \Rightarrow \mathrm{const}_c$, there is

[18]The adjective 'conical' here states that this is an oplax version of a *conical* colimit, as opposed to an oplax version of a *weighted* colimit, which is what 'colimit' often means in an enriched context (see Appendix B).

a unique morphism $u\colon$ oplax colim$F \to c$ such that the following commutes in $\underline{C}$:

$$\begin{array}{ccc} & & c \\ & \overset{\alpha_j}{\nearrow} & \uparrow{\scriptstyle u} \\ Fj & \xrightarrow[\iota_j]{} & \text{oplax colim } F. \end{array}$$

Also, for any other oplax cocone with the same tip, $\alpha'\colon F \Rightarrow \mathrm{const}_c$, which factors through oplax colim$F$ via $u'\colon$ oplax colim$F \dashrightarrow c$, as above, such that for all $j \in \mathbf{J}$, we have $\alpha'_j \Rightarrow \alpha_j$, then it must be the case that $u' \Rightarrow u$ and vice versa.

We will observe that this notion requires some adaptation for our purposes, but first we begin to analyse framed contraction, starting with its recursive case as before.

3.2.2 *Framed contraction recursive case.*

The primary use case for contraction is to compute colimits of (commuting) diagrams in $\mathrm{Zig}^{n+1}(C)$ of the form $r_0 \to s_0 \leftarrow \cdots \to s_i \leftarrow r_{i+1}$. These can be seen as a one-dimensional space indexed by regular/singular levels. Recall that, by Lemma 2.4, these correspond to a larger diagram in $\mathrm{Zig}^n(C)$ where each regular/singular level takes on the shape of a zigzag, and each zigzag map is replaced with its components; the resulting diagram can be seen as a two-dimensional space, where the points are indexed by a pair of regular/singular levels. Moreover, the original diagram is of the form of the explosion of a single point $Z \in \mathrm{Zig}^{n+2}(C)$. In general, the ($n$-fold iterated) full explosion of a zigzag in $\mathrm{Zig}^n(C)$ produces an n-dimensional space whose points are objects in C, indexed by n-tuples of regular/singular levels, connected by C-morphisms. This holds completely analogously in the case of framed zigzags also, except commuting diagrams are replaced by oplaxly commuting diagrams; note that explosion preserves the property of being an oplaxly commuting diagram, and the iterated cospan shape above is oplaxly commuting (trivially).

However, the explosion of an oplaxly commuting diagram in framed zigzags does not preserve the property of having only trivial (invertible) 2-cell fillers. This is most easily seen by the explosion of a framed zigzag map $Z \xrightarrow{h} Z'$ in some $\underline{\mathrm{Zig}^{n+1}(C)}$: the resulting 'two-dimensional' diagram in $\underline{\mathrm{Zig}^n(C)}$ has potentially non-trivial 2-cell fillers originating from the explosion of h. Similarly, the explosion of an identity morphism $Z = Z$ in $\underline{\mathrm{Zig}^{n+2}(C)}$ admits a diagram whose 2-cell fillers are all trivial, but regular-singular components are *not identities* in general (only the regular-regular and singular-singular components are). Exploding again yields a three-dimensional diagram in $\underline{\mathrm{Zig}^n(C)}$ which will admit non-trivial 2-cell fillers in general, arising from the explosion of these non-identity framed zigzag maps. If the original diagram does have some trivial 2-cell fillers, this does manifest in the exploded diagram as triviality of associated 2-cell fillers which are 'axis-aligned': just like how the explosion of an identity morphism has non-identity morphisms as regular-singular components in general, the explosion of an identity 2-cell filler in general has non-identity component 2-cell fillers relating morphisms of regular-singular variety. E.g. in Figure 5, if the 2-cell filler of framed zigzag maps $=$ inverts, then so do its axis-aligned associated 2-cell fillers in corresponding exploded diagram ($\alpha\colon r_i r''_k \Rightarrow r_i r'_j r''_k$ and $\beta\colon r_i r''_k \Rightarrow r_i s'_j r''_k$ are not axis-aligned, because they relate a regular-singular component to a composite of a regular-singular component).

Given that contraction is supposed to be a universal zigzag shortening operation, which is only rearranging information in its input diagram, in the framed setting the direct analogue is asking for a conical colimit of the same form as Figure 2. That is, all the 2-cell fillers should be trivial at the level of $\underline{\mathrm{Zig}^n(C)}$, which operationally makes sense as framed contraction should not be constructing any new algebraic information. Through the colimit lifting machinery we developed in Section 2.1, this results in (repeatedly) asking for some *interpolation* between conical and oplax conical colimit in the fully exploded diagram in $\underline{C}$, as the triviality of the 2-cell fillers in $\underline{\mathrm{Zig}^n(C)}$ yield a mix of trivial and non-trivial 2-cell fillers via this full explosion. We define this interpolation as *marked* colimits.

Definition 3.13 (Marking). A *marking* $\mathbf{M}$ of F is a class of $\mathbf{J}$-morphisms closed under composition (containing all identities).

Definition 3.14 (Marked cocone). Given a marking $\mathbf{M}$ of F, a $\mathbf{M}$-*marked cocone* over $c \in \underline{C}$ is an oplax cocone over c such that for each morphism $f\colon j \to j'$ in $\mathbf{M}$, the lax naturality triangle at f commutes strictly (i.e. its 2-cell filler inverts). $\mathbf{M}$-marked cocones over a diagram F with a fixed tip c are partially-ordered pointwise, and form a poset $\mathbf{MarkedCocone}^{\mathbf{M}}(F,c)$.

Definition 3.15 (Marked colimit). Given a marking $\mathbf{M}$ of F, the $\mathbf{M}$-*marked colimit* of F is a representing object $\mathrm{colim}^{\mathbf{M}} F$ of the $\mathbf{Pos}$-functor $\mathbf{MarkedCocone}^{\mathbf{M}}(F,-)\colon \underline{C} \to \mathbf{Pos}$.

It is the universal $\mathbf{M}$-marked cocone analogously to Definition 3.12.

The minimal marking (containing only identities) gives the oplax conical colimit, and the maximal marking (all morphisms) gives the conical colimit.

Remark 3.16. Definition 3.15 is a particular weighted colimit, just like conical and oplax conical colimits. See Appendix B for details.

To complete our analysis of the recursive nature of framed contraction, the final missing piece is an analogue of Proposition 2.3 for these marked colimits. We presuppose the notion of left Kan extension of $\mathbf{Pos}$-functors [17, § 4].

Definition 3.17 (Marked under oplax cocone 2-poset). Let $\underline{P}$ be an oplax 2-poset, generated by a simple directed acyclic graph J, along with a set $\mathbf{M}$ of edges of J. The $\mathbf{M}$-*marked under oplax cocone 2-poset* $\underline{P}^{\triangleright,\mathbf{M}}$ is given by freely adjoining to J a terminal vertex 1 along with new edges $(j,1)$ for all $j \in J$, and taking the quotient of its free oplax 2-poset which inverts $j1 \Rightarrow jj'1$ for each $(j,j') \in \mathbf{M}$; it is equipped with an inclusion $i\colon \underline{P} \hookrightarrow \underline{P}^{\triangleright,\mathbf{M}}$.

1 is the maximal element of $\Pi_{0*}(\underline{P}^{\triangleright,\mathbf{M}})$, and every Hom poset $\underline{P}^{\triangleright,\mathbf{M}}(j,1)$ admits a meet given by the atom $j1$.

Proposition 3.1. *For any oplaxly commuting diagram $F\colon \underline{P} \to \underline{C}$, the data of an $\mathbf{M}$-marked colimit of F is equivalent to the left Kan extension* $\mathrm{Lan}_i F$*:*

$$\begin{array}{ccc} \underline{P} & \xrightarrow{F} & \underline{C} \\ {\scriptstyle i}\downarrow & \nearrow_{\mathrm{Lan}_i F} & \\ \underline{P}^{\triangleright,\mathbf{M}} & & \end{array} \qquad \textit{given by} \quad \mathrm{Lan}_i F(j) = \begin{cases} Fj, & j \in \underline{P}, \\ \mathrm{colim}^{\mathbf{M}} F, & j = 1, \end{cases}$$

and $\mathrm{Lan}_i\, F(f) = Ff$ *for morphisms of* $\underline{P}$, $\mathrm{Lan}_i\, F(j_0 \dots j_k 1) = \iota_{j_k} \circ F(j_0 \dots j_k)$ *for each new morphism in* $\underline{P}^{\triangleright,\mathbf{M}}$.

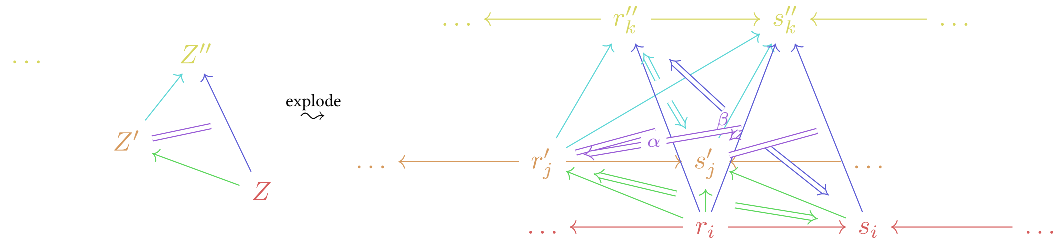


Figure 5: Explosion of framed zigzags. Colour indicates originating framed zigzag/framed zigzag map/2-cell filler.

Now, the translation machinery of Section 2.1 works essentially unchanged, because:

- marked, oplax conical, and conical colimits in $D_*(\Delta_+)$ coincide because its Hom posets are discrete,
- the parametric right adjoint $\underline{\text{Zig}}: \mathbf{PosCat} \to \mathbf{PosCat}$ extends to a strict 2-functor in the same way as Proposition 2.1,
- $\mathbf{PosCat}$ and $\mathbf{PosCat}/\underline{C}$ have a completely analogous 2-categorical structure to $\mathbf{Cat}$ and $\mathbf{Cat}/C$, and
- $\text{dom}: \mathbf{PosCat}/\underline{C} \to \mathbf{PosCat}$ is also locally fully faithful.

Therefore, Theorem 2.10 and Corollary 2.4 hold for $\underline{\text{Zig}}$ and marked colimits instead.

3.2.3 Framed contraction base case. Finally, we need to construct some $\mathbf{M}$-marked colimit of $F: D(C(J)) \to \underline{C}$, with F a diagram obtained by explosion. As we did in Section 2.1.3, we assume that such a diagram is surjective without loss of generality.

We note that this diagram is related to its unframed version in a strong way.

Lemma 3.3. *If F admits a marked colimit $\text{colim}^{\mathbf{M}} F$ for any $\mathbf{M}$, $\Pi_{0*}(F): \Pi_{0*}(D(C(J))) \to \Pi_{0*}(\underline{C})$ admits a colimit given by the same object whose legs are given by applying Π_{0*}.*

This means we can determine the tip of the universal $\mathbf{M}$-marked cocone whenever it exists by first unframing and then taking the colimit in the poset $\Pi_{0*}(\underline{C})$. By surjectivity, this colimit takes the form $F(j_\top)$ for some $j_\top \in T$, where T is the set of terminal vertices of J; let $T^{\mathbf{M}}$ denote the subgraph of J induced by T and marked edges, which is connected by axis alignment. The only remaining information is to determine its legs ι_j:

(1) the $\iota_{j_\top}$ is given by $\text{id}_{F(j_\top)}$ necessarily, because every endomorphism in $\underline{C}$ is identity;
(2) because every marked edge imposes a commuting triangle requirement with respect to the tip $F(j_\top)$, we can progressively determine the legs at every vertex in $T^{\mathbf{M}}$: for an edge jj' in $T^{\mathbf{M}}$, if $\iota_{j'}$ is known, then ι_j must be given by composition as $\iota_{j'} \circ F(jj')$; similarly, if ι_j is known, and $F(jj')$ is not an identity, then this commutativity condition is unsatisfiable[19], so fail; if it is an identity, then set $\iota_{j'} = \iota_j$;
(3) now, extend this to the rest of J: for each jj' with $j' \in T$, $\iota_j = \iota_{j'} \circ F(jj')$ satisfies the oplaxly commuting triangle above jj'; every 2-cell filler in the neighbourhood of jj' in the image of F determines a constraint system of inequalities of the poset $\underline{C}(F(j), F(j'))$ of alternative choices for $F(jj')$ which would still make ι_j an $\mathbf{M}$-marked cocone — fail if there is ever a distinct choice other than $F(jj')$, as in that situation we will be able to construct two distinct $\mathbf{M}$-marked cocones with the same tip, which necessarily must factor through each other by $\text{id}_{F(j_\top)}$, a contradiction.

The resulting $\mathbf{M}$-marked cocone is the universal one (in fact, the only one), as we have checked along the way, and this procedure terminates as we always make progress along the finitely many vertices of J.

[19] Like how the composite of non-identity morphisms in an oplax 2-poset is never atomic.

4 Coherent invertibility

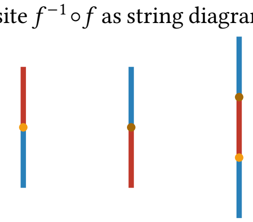
Here we demonstrate some examples [15, Chapter 6] of framed zigzag $\mathbf{Pos}$-categories which model the setting for n-dimensional string diagrams, with respect to some algebraic signature, which admit coherently invertible cells. For lack of a better scheme, morphisms in the base $\mathbf{Pos}$-category (which represent framing) are simply indexed by positive natural numbers, but the reader is reminded that they are supposed to signify *proof-relevant boundary inclusion* of generators in the context of an n-dimensional string diagram.

Each example is constructed inside homotopy.io, and can be easily reproduced by the interested reader[20]. Some pre-constructed workspaces are available at the referenced links. Further guidance on the use of the system is available [19, 12, 5].

4.1 Invertible 1-cell [14]

The algebraic signature to be modelled consists of two 0-cells x and y, and a single 1-cell $f: x \to y$ which admits an inverse $f^{-1}: y \to x$.

The base $\mathbf{Pos}$-category in this case is the $\mathbf{Pos}$-category $\underline{C}$ given by the diagram $x \xrightarrow{1} f \xleftarrow{2} y$. 1-dimensional string diagrams over this signature are encoded by objects of $\underline{\text{Zig}}(\underline{C})$ such as $x \xrightarrow{1} f \xleftarrow{2} y$, $y \xrightarrow{2} f \xleftarrow{1} x$, and $x \xrightarrow{1} f \xleftarrow{2} y \xrightarrow{2} f \xleftarrow{1} x$, which respectively represent f, f^{-1} and the composite $f^{-1} \circ f$ as string diagrams:

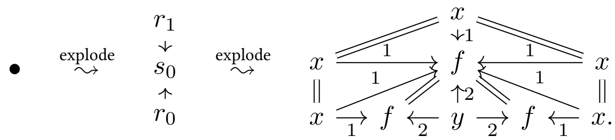


Here the coloured wires represent x and y, with f and f^{-1} being coloured points. In our string diagram convention, diagrams go from bottom-to-top.

[20] In order for the system to allow the inverse of an n-cell for $n > 1$ to be attached, that cell must be marked 'invertible' inside the signature.

The cancellation $f^{-1} \circ f \Rightarrow \mathrm{id}_x$ seen as an invertible 2-cell is represented by the 2-dimensional string diagram:

This is encoded by some object $\bullet \in \underline{\mathrm{Zig}^2(C)}$, with explosions in $\underline{\mathrm{Zig}(C)}$ and $\underline{C}$[21]:

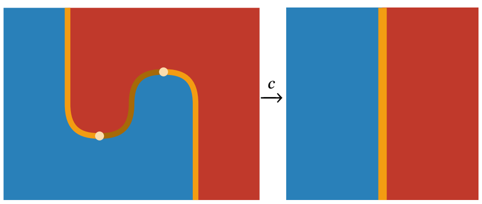


In particular, this diagram is obtained by contracting the 1-dimensional string diagram $f^{-1} \circ f$, and observe that f is the (marked) colimit of the directed diagram:

$$x \xrightarrow{1} f \xleftarrow{2} y \xrightarrow{2} f \xleftarrow{1} x,$$

as we described in Section 3.2.3, and this determines the framed zigzag map $r_0 \to s_0$ above.

An example witness to f^{-1} being a *coherent* inverse to f is given by the 'snakerator' framed zigzag map of the following form:



Crucially, such a framed zigzag map is obtained by framed contraction, and we conjecture that all such coherences in every dimension are also obtainable this way.

This can also be seen as part of a 3-cell, which is the 3-dimensional string diagram:

The front face of this picture is encoded by the domain of c, and the back face by its codomain. Scanning continuously front-to-back, c itself encodes the homotopical deformation of 'straightening the snake' — that the transformation in Equation (1) is equivalent to the 'do nothing' (identity) transformation. This 3-cell is itself coherently invertible, and will admit an infinite family of coherences in the same way as f. More higher coherences are detailed in Appendix C.

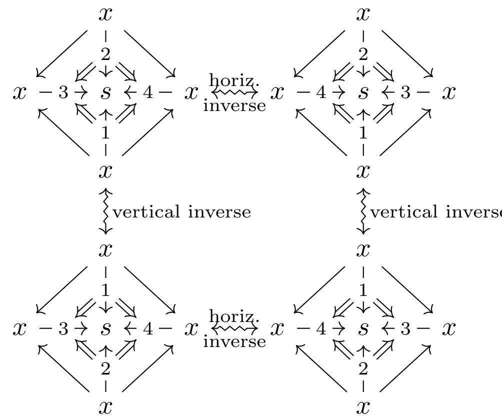
4.2 Invertible 2-cell scalar

In this example, we model the algebraic signature consisting of a single 0-cell x, and a single 2-cell scalar $s\colon \mathrm{id}_x \to \mathrm{id}_x$ which admits vertical and horizontal inverses of the same type.

The base **Pos**-category in this case is $\underline{C}$ given by two objects x and s, with the only non-identity morphisms being:

$$C(x,s) := \begin{matrix} 3 \\ | \\ 1 \end{matrix} \times \begin{matrix} 4 \\ | \\ 2 \end{matrix}$$

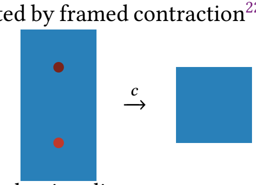
The objects of $\underline{\mathrm{Zig}^2(C)}$, drawn as diagrams in $\underline{C}$ by explosion, which represent s and its inverses are:



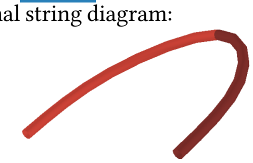
The top-left diagram represents s itself, with its vertical inverse s^{-1} being represented by its reflection in the horizontal axis below, and its horizontal inverse being represented by its reflection in the vertical axis to the right.

Even though as diagrams in $\underline{C}$ these all appear to contain the same data, as objects of $\underline{\mathrm{Zig}^2(C)}$ they are distinct. This example also illustrates the necessity of **Pos**-enrichment and non-trivial 2-cell fillers: without that, commutativity of all the triangles would be imposed, which in turn determines $1=2=3=4$ and removes the ability for s to be distinguished from its inverses in $\underline{\mathrm{Zig}^2(C)}$.

The cancellation of $s^{-1} \circ s = \mathrm{id}_{\mathrm{id}_x}$ is again given by the framed zigzag map generated by framed contraction[22]:



or as a 3-dimensional string diagram:



That this inverse is coherent says, among other things, that the 3D snake that can be constructed by extending this also admits a homotopy obtained by contraction which pulls it straight.

In Appendix D we use an invertible scalar to illustrate a formal proof of Theorem 1.1, as stated in the introduction.

[21] All 2-cell fillers in $\underline{C}$ are trivial, so have been elided in the diagram.

[22] See Appendix E to see how all the pieces fit together.

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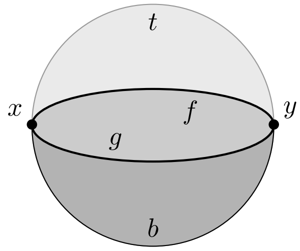
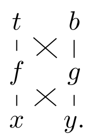


Figure 6: CW decomposition of the 2-sphere.

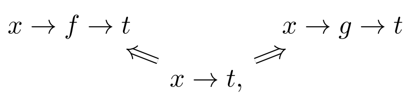
A Face poset and entrance path Pos-category

Here we give an example of the entrance path **Pos**-category (i.e. 2-poset) associated to a CW decomposition of space, which is a categorification of its face poset. This structure tracks *proof-relevant* boundary inclusion in an analogous way to how framed zigzag **Pos**-categories generalise the original zigzag categories of Reutter and Vicary [20] by tracking framing data.

Consider the CW decomposition in Appendix 6 [18, Figure 3]. Its face poset P is given by cells ordered by boundary inclusion, i.e.



However, information on *how* a cell includes as the boundary of another is lost, because the face poset is thin. The entrance path **Pos**-category $\underline{P}$ of this CW decomposition restores this information; it has the same cells as objects, but rather than a boolean-valued Hom between cells, it has a poset of *entrance paths* ordered by the subsequence relation: e.g. the poset $\underline{P}(x,t)$ is



which represents that there are two ways to *enter* t from x, either through f or g, and neither is a refinement of the other. We observe the relationship between the two structures: $P = \Pi_{0*}(\underline{P})$.

Similarly, we utilise **Pos**-categories to 'thicken' the Homs of the category encoding an algebraic signature at the base of the zigzag construction (a poset) into **Pos**-categories in the framed zigzag construction to allow for the representation of proof-relevant boundary inclusions, which we call framing.

B Marked colimits are weighted colimits

In this section, we elaborate on the well-known fact that oplax conical colimits are weighted colimits for a certain weight, and extend this description to encompass marked colimits. This means that they exist in any cocomplete **Pos**-category (which admit all weighted colimits). The original result is due to Street [23], who showed it in the case of 2-categories; we give a more explicit form. First, we recall the notion of weighted colimit for **Pos**-categories.

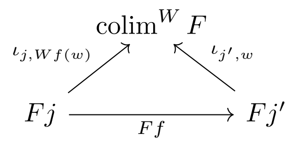
*Definition B.1 (**Pos**-weighted colimit).* A weighted colimit of a diagram $F: \mathbf{J} \to \underline{C}$ weighted by $W: \mathbf{J}^{\text{op}} \to \mathbf{Pos}$, $\text{colim}^W F$, is an object of $\underline{C}$ with **Pos**-representation

$$\underline{C}\left(\text{colim}^W F, c\right) \cong [\mathbf{J}^{\text{op}}, \mathbf{Pos}]\left(W, \underline{C}(F-, c)\right).$$

Explicitly, this equips $\text{colim}^W F$ with a family of morphisms

$$\left(\iota_{j,w}: Fj \to \text{colim}^W F\right)_{j \in \mathbf{J}, w \in Wj}$$

such that for all $f: j \to j'$ and $w \in Wj'$:



commutes, and this is universal: any other object with such a family of morphisms factors uniquely via $\text{colim}^W F$. This factorisation and the mapping $Wj \to \underline{C}(Fj, \text{colim}^W F)$ are required to be monotone.

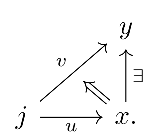
Remark B.2. W is a generalisation of the shape of the cocone. A natural transformation $\alpha_j: Wj \to \underline{C}(Fj, c)$ is a collection of maps which act as legs of this generalised cocone, varying monotonically over weights $w \in Wj$; the naturality property asserts that for any $f: j \to j'$, varying the weight determined by a specific leg $Fj' \to c$ by mapping along Wf is equal to precomposing that leg by Ff — in other words, the previous triangle commutes replacing c for $\text{colim}^W F$. The order isomorphism between the poset of these natural transformations and the Hom poset $\underline{C}(\text{colim}^W F, c)$ then establishes the universal property: firstly that a morphism $\text{colim}^W F \to c$ is in bijective correspondence with a particular generalised cocone of tip c, and that moreover this bijection should preserve and reflect orders[23]. That is, the order on factoring maps $\underline{C}(\text{colim}^W F, c)$ is completely determined by cocones, one above another exactly when the target cocones are ordered as such.

The ordinary (conical) colimit is obtained by taking W to be the constant functor at the terminal object of **Pos**.

Proposition B.1 (Oplax conical colimits are weighted colimits [23]). *Every oplax conical colimit of $F: \mathbf{J} \to \underline{C}$ is equivalently given by a colimit of F weighted by some $W: \mathbf{J} \to \mathbf{Pos}$.*

The explicit description of this transformation is given as follows.

Definition B.3 (Oplax conical weight). Let $F: \mathbf{J} \to \underline{C}$ be a diagram. The *oplax conical weight* of F is the contravariant **Pos**-functor $L_{\mathbf{J}}: \mathbf{J}^{\text{op}} \to \mathbf{Pos}$ which sends j to the set of morphisms in $\mathbf{J}$ with domain j, ordered by $u \le v$ whenever there exists some $\mathbf{J}$-morphism which oplaxly extends u to v:



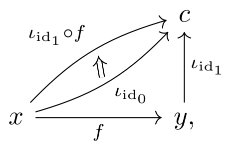
A $\mathbf{J}$-morphism $f: j \to j'$ is sent the monotone map of precomposition with f, and Hom posets inherit the ordering on $\mathbf{J}$-morphisms.

Example B.4. Let $\mathbf{J}$ be the walking arrow $0 \to 1$, and $F: \mathbf{J} \to \underline{C}$ the diagram which chooses $f: x \to y$.

Its oplax conical weight $L_{\mathbf{J}}$ is given by $0 \mapsto \{\text{id}_1\}$ and $1 \mapsto \{\text{id}_0 \le 0 \to 1\}$, with the unique morphism $0 \to 1$ mapping to the monotone map $- \circ (0 \to 1)$. A colimit of F weighted by $L_{\mathbf{J}}$ is a universal object c,

[23]Recall that the natural transformations α are ordered pointwise by components.

two legs $\iota_{\mathrm{id}_0} \Rightarrow \iota_{0\to 1}$ over $F(0) = x$, one leg ι_{id_1} over $F(1) = y$, satisfying $\iota_{0\to 1} = \iota_{\mathrm{id}_1} \circ F(0 \to 1) = \iota_{\mathrm{id}_1} \circ f$. Pictorially, this looks like:



which is exactly what it means for ι_{id_0} and ι_{id_1} to form an oplax cocone over F.

Now, we extend this to marked colimits.

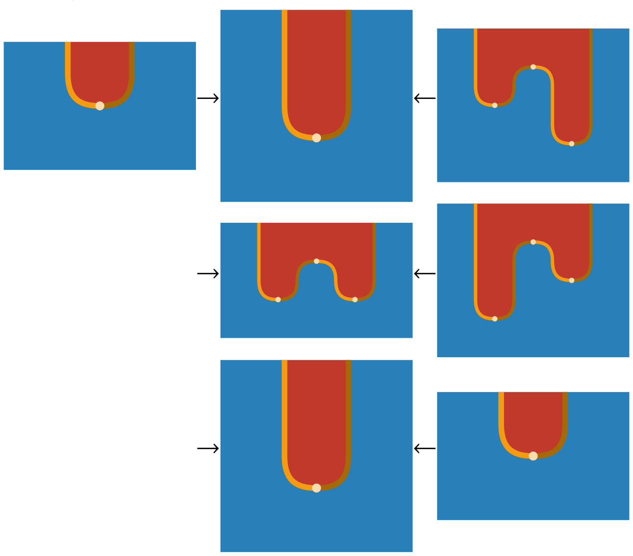
Definition B.5 (Marked weight). The *marked weight* $\mathbf{M}_\mathbf{J}$ associated to a **Pos**-functor $F : \mathbf{J} \to \underline{C}$ equipped with a marking $\mathbf{M}$ is defined similarly to the oplax conical weight $L_\mathbf{J}$, except each poset in its image is additionally localised at

$$\{\mathrm{id}_x \leq f : \exists y \in \mathbf{J}. f : x \to y \in \mathbf{M}\}, \quad \text{for each } x \in \mathbf{J}.$$

The fact that $\mathbf{M}$ is closed under composition ensures that $\mathbf{M}_\mathbf{J}$ is well-defined as a **Pos**-functor (on morphisms).

C Catastrophe: snakes, swallowtails, and butterflies

Continuing from Section 4.1, one dimension higher, *swallowtail* coherences assert that certain 3-cells built from the introduction and elimination of the snake along snakerator 2-cells are equal. For instance, the assertion that



is equivalent to the identity 3-cell. From the third picture, there are two ways to reach the 'cup' diagram: perform a snake cancellation move on the left, or interchange the cups, and then perform a snake cancellation move on the right. The swallowtail coherence asserts that these are equivalent.

It is a 4-cell itself, so it is difficult to convey in static images, but homotopy.io allows us to represent this fourth dimension as a smooth animation of 3-dimensional surface diagrams. Figure 7 presents screenshots of 3D time slices of this animation.

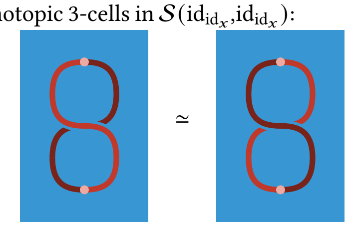
This admits a 5-dimensional coherence: the *butterfly* coherence, asserting that certain sequences of transformations of 3-dimensional surface diagrams built from swallowtail introduction/cancellation moves and interchange are equivalent to the 'do nothing' transformation. The source of this 5-cell (a 4-cell) can be viewed as an animation, as in Figure 8 (the target 4-cell is a static animation).

All of these coherences can be interactively explored in the linked workspace [14]. Together, these coherences are call *catastrophes*, and we conjecture that the mechanism of framed contraction can generate them in arbitrary dimensions.

D The Figure-8 Theorem [16]

Here we illustrate a formalised proof of Theorem 1.1.

Theorem 1.1. In the free ∞-groupoid $\mathcal{S}$ generated by an object x and a 2-cell $f : \mathrm{id}_x \Rightarrow \mathrm{id}_x$, the following 'Figure-8' string diagrams represent homotopic 3-cells in $\mathcal{S}(\mathrm{id}_{\mathrm{id}_x}, \mathrm{id}_{\mathrm{id}_x})$:



This proof can also be viewed as an animation of 3-dimensional string diagrams [13].

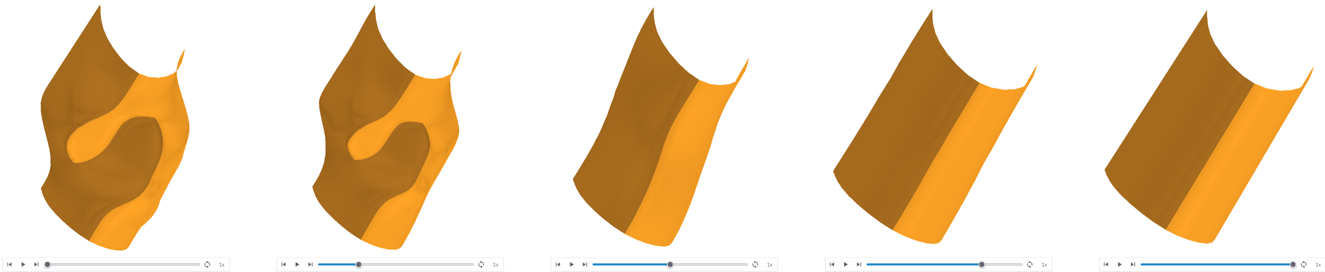


Figure 7: Swallowtail coherence, rendered at time $t = 0, 0.25, 0.5, 0.75, 1$.

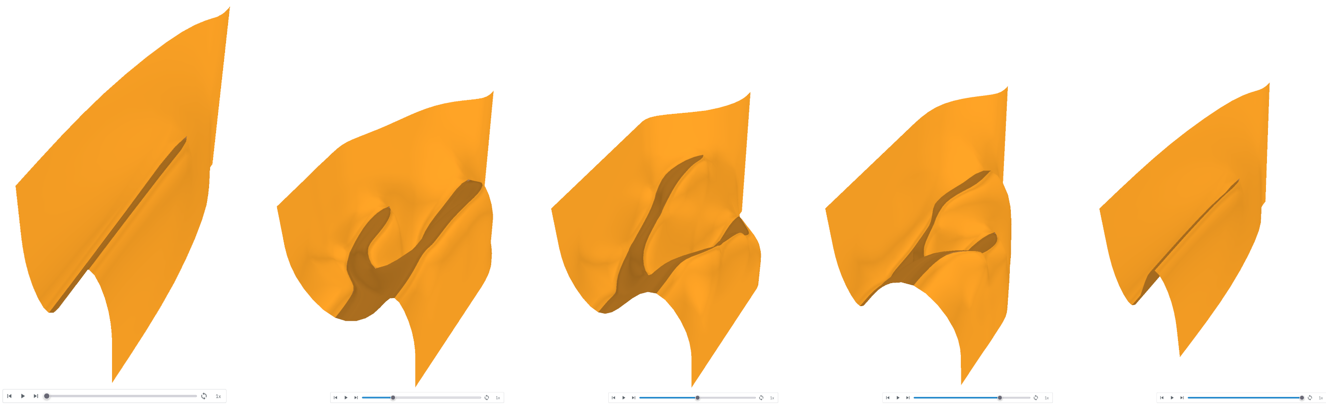
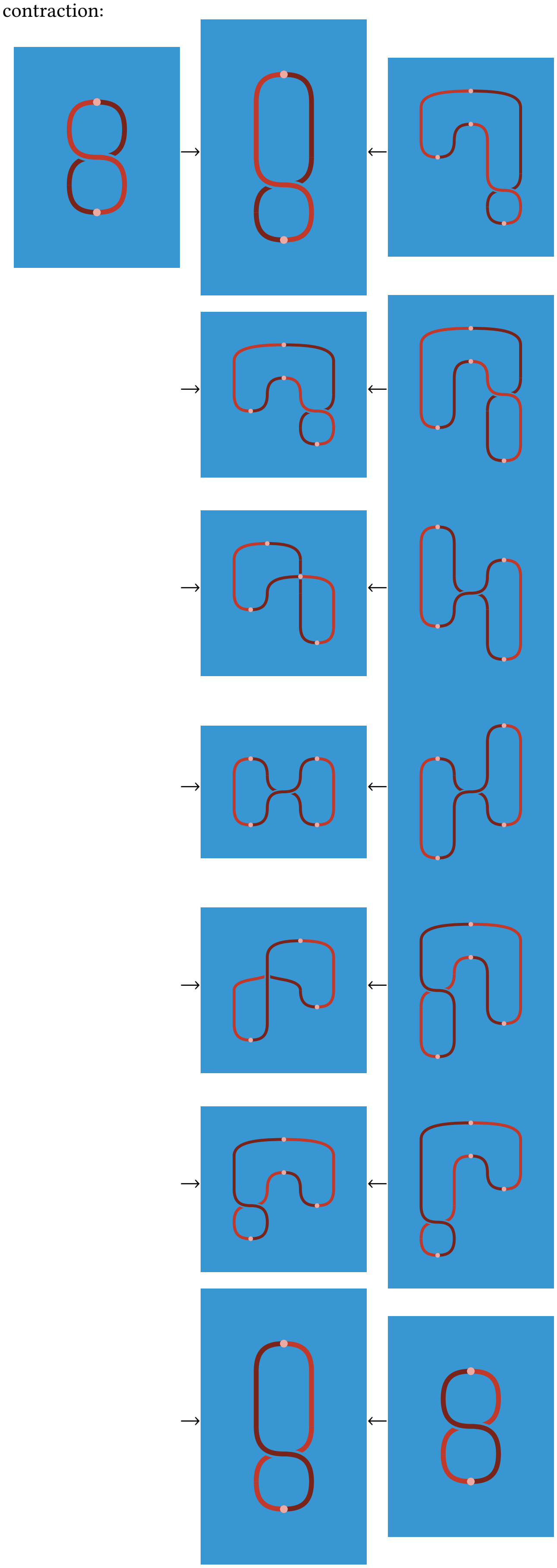


Figure 8: Butterfly coherence source, rendered at time $t = 0, 0.25, 0.5, 0.75, 1$.

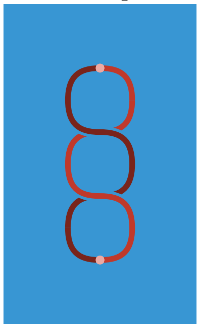
Proof [15]. By string diagram homotopy constructed by framed contraction:



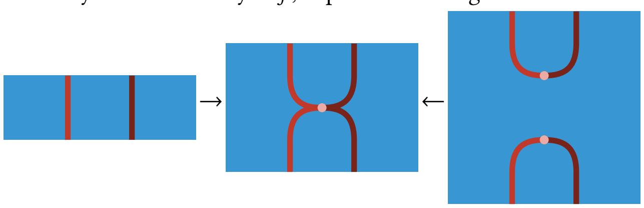
□

Corollary D.1. $\pi_3(S^2) \cong \mathbb{Z}$.

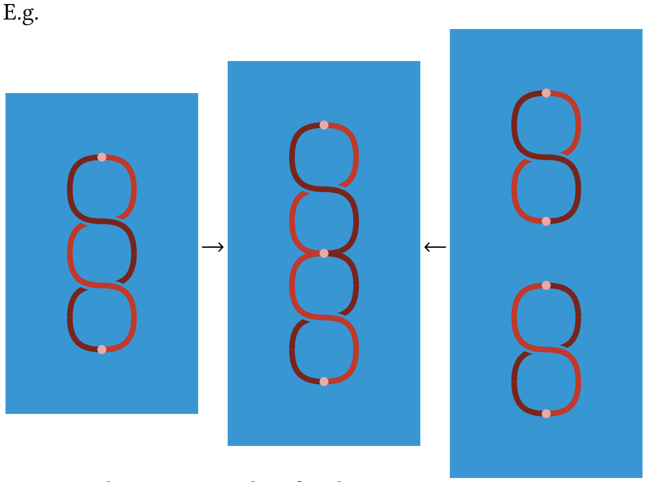
Proof [15]. In string diagrams, consider the set of generators of $\pi_3(S^2)$ corresponding to some (composite) braiding, e.g.



Such a generator can be decomposed into a composition of Figure-8 3-cells by the invertibility of f, in particular along:



E.g.



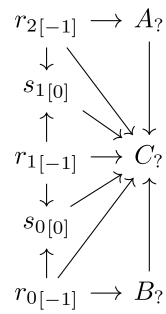
Any such generator is therefore homotopic to some composition of Figure-8s, using the fact that there are a priori four possible Figure-8 3-cells arranged into two pairs of inverses, both of which coincide (Theorem 1.1).

Note that this set of generators extends to all of $\mathcal{S}(\mathrm{id}_{\mathrm{id}_{\mathrm{id}_x}}, \mathrm{id}_{\mathrm{id}_{\mathrm{id}_x}})$ by homotopy along the snake homotopies admissible by the invertibility of f, so we deduce that $\pi_3(S^2)$ is freely generated by a single non-trivial generator so it is isomorphic to the integers under addition: $\mathbb{Z} := (\{0,1,-1\},+,0)$. □

E A worked example of recursive framed contraction

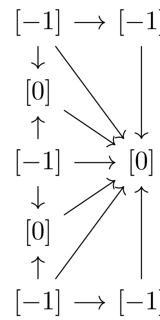
Here, we elaborate on the recursive case of the framed contraction procedure involved in the contraction of Section 4.2, using Section 4.2 as our example.

(1) First, set up the conical colimit framed contraction is to compute (Section 3.2.2):

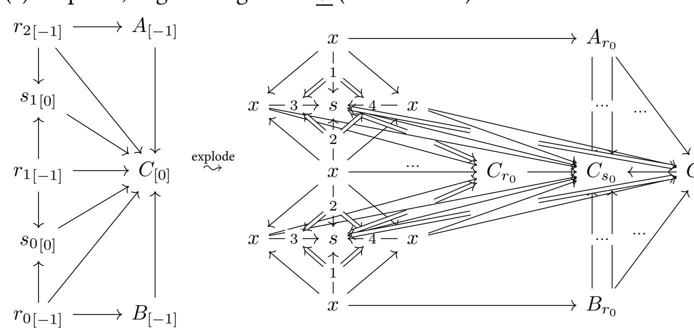


The ?s indicate unknown shapes for the result framed zigzags, to be determined.

(2) Take the singular projection, and compute the colimits in Δ_+ to determine those shapes (Section 2.1.1):

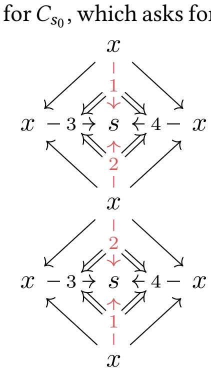


(3) Explode, to get a diagram in $\underline{C}$ (Section 2.1.2):



(many arrows and 2-cell fillers have been omitted for readability).

(4) This induces a collection of marked colimit problems in $\underline{C}$ to determine both what the $\underline{C}$-objects $A_{r_0}, B_{r_0}, \ldots$ should be, and the legs of their respective marked cocones (Section 3.2.3); the most interesting one is for C_{s_0}, which asks for a marked colimit of:



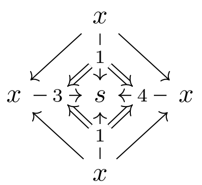
where red arrows indicate marking, induced by explosion from the axis-aligned trivial 2-cell fillers in $\underline{\mathrm{Zig}(C)}$ that we started with.

(5) Because we corestrict to make this diagram surjective (Section 2.1.3), via unframing (Section 3.1.1) surmise that the tip of this marked colimit is necessarily s.

(6) Through the base case procedure in Section 3.2.3:
   (a) determine the legs from top to bottom of the vertically aligned xs as 1, 2, and 1 respectively;
   (b) for the xs on the left, determine the constraints in $\underline{C}(x,s)$ to be ≤ 3, ≥ 1, ≥ 2: the only solution is 3; symmetrically, determine the legs for the xs on the right to be 4.

   These give part of the data for the maps into $C_{[0]}$ that we need to compute.

(7) Repeat for the other marked colimit problems, finally yielding the result as:



along with all the data mapping in, required to assemble the framed zigzag maps in $\underline{\mathrm{Zig}(C)}$ from step (3).

F Proofs

Here we include proofs that have been elided in the main text for the sake of readability.

F.1 Proofs for Section 2 (Zigzag categories)

Lemma 2.1 (Wraith's duality [28]). *There is an equivalence of categories* $\mathrm{R}\colon \Delta_+ \cong \Delta_=^{\mathrm{op}}$ *given by* $[n] \mapsto [n+1]$.

Proof. To complete the proof rigorously, the equivalence R is given by retyping (in the codomain) the contravariant Hom functor as

$$\Delta_+(-,[2])\colon \Delta_+ \xrightarrow{\sim} \Delta_=^{\mathrm{op}},$$

using the fact that $[n+1]$ can be identified with the set of monotone maps $\alpha\colon [n] \to [2]$. This is because any such monotone map α is uniquely determined by $i \in [n+1]$ such that $\forall j \le i. \alpha(j) = 0$. In other words, thinking of $[n]$ as an iterated cospan as above, α is determined by choosing an r_i such that it partitions the s_j on the left into those mapped to 0 and on the right into those mapped to 1, of which there are $[n+1]$ choices. It follows easily that such a functor is fully faithful and essentially surjective. □

Corollary 2.1. *Every category C admits a unique functor to the terminal category* $!\colon C \to \mathbf{1}$. *The singular projection π_C factors as the composite* $\pi_1 \circ \mathrm{Zig}(!)$.

Proof. Observe that the isomorphism of Lemma 2.2 is precisely π_1. The necessary commuting triangle arises from the functoriality of Zig (Lemma 2.3). □

Lemma 2.4 (Universal property of zigzag categories). *We have an adjunction as follows:*

$$\mathbf{Cat} \underset{\mathrm{Zig}}{\overset{\mathrm{Expl}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Cat}\,/\,\boldsymbol{\Delta}_+\,.$$

Proof. We determine a natural bijection between Δ_+-indexed diagrams in a zigzag category $\mathrm{Zig}(C)$ and diagrams in C with respect to an exploded index:

$$\frac{F\colon \mathbf{J} \to \mathrm{Zig}(C) \qquad \text{in } \mathbf{Cat}/\Delta_+,}{\int F\colon \mathrm{Expl}(\pi_C \circ F\colon \mathbf{J} \to \Delta_+) \to C \qquad \text{in } \mathbf{Cat}.}$$

$\int F$ is the functor which maps on objects, for each $x \in \mathbf{J}$ over $[m]$,

$$x_{[m]}(r_i) \mapsto Fx(r_i), \qquad 0 \le i \le m+1,$$
$$x_{[m]}(s_j) \mapsto Fx(s_j), \qquad 0 \le j < m+1,$$

and on morphisms, for $h \in \mathbf{J}(x,y)$ over $\alpha\colon [m] \to [n]$,

$$h_\alpha(r_{i'}) \mapsto Fh(r_{i'}), \qquad 0 \le i' \le \mathrm{R}[n] = n+1,$$
$$h_\alpha(s_j) \mapsto Fh(s_j), \qquad 0 \le j < m+1$$
$$h_\alpha(r_i, s_{j'}) \mapsto Fh(r_i, s_{j'}), \qquad \begin{array}{c} 0 \le j' < n+1, \\ \mathrm{R}\alpha(j') \le i \le \mathrm{R}\alpha(j'+1). \end{array}$$

The inverse of this bijection is analogous, and naturality is straightforward. □

Proposition 2.1. Zig *extends to a strict 2-functor.*

Proof. Fix functors $F,G\colon C \to \mathcal{D}$. Recall that

$$\mathrm{Zig}(F), \mathrm{Zig}(G)\colon \mathrm{Zig}(C) \to \mathrm{Zig}(\mathcal{D})$$

are Δ_+-indexed functors which reannotate zigzags in C to $\mathcal{D}$, which are compatible with the singular projection: they leave the shape of zigzags/zigzag maps unchanged.

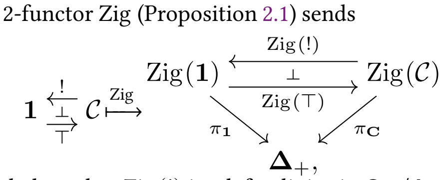
We need to define the action of Zig on natural transformations $\alpha\colon F \Rightarrow G$, and its result must be a Δ_+-indexed natural transformation $\mathrm{Zig}(\alpha)\colon \mathrm{Zig}(F) \Rightarrow \mathrm{Zig}(G)$. Strictness of $\mathbf{Cat}/\Delta_+$ means that this is merely a natural transformation between the underlying functors via the domain projection. This is a collection of zigzag maps $\mathrm{Zig}(\alpha)_Z\colon \mathrm{Zig}(F)(Z) \to \mathrm{Zig}(G)(Z)$ in $\mathrm{Zig}(\mathcal{D})$ parametrised by $Z \in \mathrm{Zig}(C)$, subject to naturality conditions. All of Z, $\mathrm{Zig}(F)(Z)$, and $\mathrm{Zig}(G)(Z)$ have the same length, so this fixes the shape of $\mathrm{Zig}(\alpha)_Z$: its underlying monotone map is the identity function. Now use the components of α as the required $\mathcal{D}$-morphisms; the validity of this zigzag map is given by the naturality of α. Naturality of $\mathrm{Zig}(\alpha)_Z$ is easy to check. □

Proposition 2.2. *A category C admits a terminal object if and only if the terminal functor* $!\colon C \to \mathbf{1}$ *admits a right adjoint.*

Proof. The right adjoint $\top\colon \mathbf{1} \to C$ is the global element picking out the terminal object of C; for any object $c \in C$, we obtain $\{*\} \cong \mathbf{1}(!c, *) \cong C(c, \top *)$. □

Corollary 2.3. *If C admits a terminal object, then the singular projection* $\pi_C\colon \mathrm{Zig}(C) \to \Delta_+$ *preserves colimits.*

Proof. Because strict 2-functors preserve adjunctions (Lemma 2.5), the strict 2-functor Zig (Proposition 2.1) sends



and thus deduce that $\mathrm{Zig}(!)$ is a left adjoint in $\mathbf{Cat}/\Delta_+$ (and hence by Corollary 2.2 a left adjoint functor). By Corollary 2.1, we deduce that $\pi_C = \pi_1 \circ \mathrm{Zig}(!)$ is the composite of left adjoints and so also a left adjoint, and hence preserves colimits. □

Lemma 2.8. *If C admits a terminal object, then so does* $\mathrm{Zig}(C)$.

Proof. Abstractly, following the proof of Corollary 2.3, we obtain an adjunction in $\mathbf{Cat}/\Delta_+$: id_{Δ_+} is left adjoint to π_C. The strict 2-functor dom sends this to an adjunction in $\mathbf{Cat}$: Δ_+ is left adjoint to $\mathrm{Zig}(C)$. Now, compose this adjunction with the adjunction between $\mathbf{1}$ and Δ_+ which determines the terminal object $[0] \in \Delta_+$ in order to get a right adjoint to $! \colon \mathrm{Zig}(C) \dashrightarrow \mathbf{1}$.

Explicitly, the terminal zigzag in $\mathrm{Zig}(C)$ is the 1-length zigzag $1_{[0]} \coloneqq \top \to \top \leftarrow \top$; every other zigzag admits a unique zigzag map to this: the shape is fixed because $[0]$ is terminal in Δ_+, and the type is fixed because $\top$ is terminal in C. □

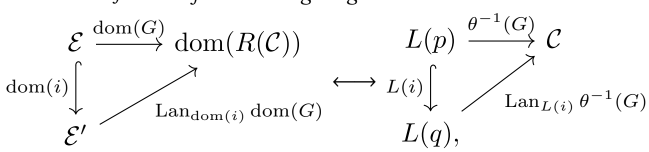
Theorem 2.10. *Let $F\colon \mathbf{Cat} \to \mathbf{Cat}$ be a parametric right adjoint endofunctor, which factors as:*

$$F\colon \mathbf{Cat} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathbf{Cat}/F\mathbf{1} \xrightarrow{\mathrm{dom}} \mathbf{Cat},$$

with the natural bijection of the adjunction denoted:

$$\forall C \in \mathbf{Cat}, p\colon \mathcal{E} \to F\mathbf{1} \in \mathbf{Cat}/F\mathbf{1}. \quad [L(p), C] \overset{\theta}{\cong} [p, R(C)]_{/F\mathbf{1}}.$$

For a $F\mathbf{1}$-indexed categories $p\colon \mathcal{E} \to F\mathbf{1}$ and $q\colon \mathcal{E}' \to F\mathbf{1}$, and $F\mathbf{1}$-indexed functors $G\colon p \to R(C)$ and $i\colon p \hookrightarrow q$, where i is an inclusion[24]*, we have a bijection of commuting diagrams in* $\mathbf{Cat}$:



i.e. $\mathrm{Lan}_{\mathrm{dom}(i)}\mathrm{dom}(G) \cong \mathrm{dom}(\theta(\mathrm{Lan}_{L(i)}\theta^{-1}(G)))$.

Proof. For any functor $H'\colon \mathcal{E}' \to \mathrm{dom}(R(C))$, by composition there is a $F\mathbf{1}$-indexed category $R(C) \circ H'\colon \mathcal{E}' \to F\mathbf{1}$ along with a $F\mathbf{1}$-indexed functor $H\colon R(C)\circ H' \to R(C)$ such that $\mathrm{dom}(H) = H'$. Now calculate:

$$\begin{aligned}
&[\mathcal{E}', \mathrm{dom}(R(C))]\,(\mathrm{Lan}_{\mathrm{dom}(i)}\mathrm{dom}(G), \mathrm{dom}(H))\\
&\cong \{\mathrm{Lan}_{\mathrm{dom}(i)} \dashv - \circ \mathrm{dom}(i)\}\\
&[\mathcal{E}, \mathrm{dom}(R(C))]\,(\mathrm{dom}(G), \mathrm{dom}(H)\circ\mathrm{dom}(i))\\
&\cong \{\mathrm{dom}\text{ locally fully faithful 2-functor}\}\\
&[p, R(C)]_{/F\mathbf{1}}(G, H\circ i)\\
&\cong \{\theta^{-1}\text{ natural bijection}\}\\
&[L(p), C]\,(\theta^{-1}(G), \theta^{-1}(H\circ i))\\
&\cong \{\theta^{-1}(x) = \varepsilon_C \circ L(x)\}\\
&[L(p), C]\,(\theta^{-1}(G), \theta^{-1}(H)\circ L(i))\\
&\cong \{\mathrm{Lan}_{L(i)} \dashv - \circ L(i)\}\\
&[L(q), C]\,(\mathrm{Lan}_{L(i)}\theta^{-1}(G), \theta^{-1}(H))\\
&\cong \{\theta\text{ natural bijection}\}\\
&[q, R(C)]_{/F\mathbf{1}}(\theta(\mathrm{Lan}_{L(i)}\theta^{-1}(G)), H)\\
&\cong \{\mathrm{dom}\text{ locally fully faithful 2-functor}\}\\
&[\mathcal{E}', \mathrm{dom}(R(C))]\,(\mathrm{dom}(\theta(\mathrm{Lan}_{L(i)}\theta^{-1}(G))), \mathrm{dom}(H)).
\end{aligned}$$

Hence, by the Yoneda embedding, deduce the result. □

[24] i.e. $\mathrm{dom}(i)\colon \mathcal{E} \hookrightarrow \mathcal{E}'$ is a subcategory inclusion.

Corollary 2.4 (Colimits in $\mathrm{Zig}(C)$ from colimits in C). *Left Kan extensions along inclusions i (which subsume colimits) in the Δ_+-indexed category $\mathrm{Zig}(C)$ are computed as left Kan extensions along $\mathrm{Expl}(i)$ in C: for $F\colon \mathbf{J} \to \mathrm{Zig}(C)$ and $i\colon \pi_C \circ F \hookrightarrow \pi_C \circ \mathrm{Lan}_{\mathrm{dom}(i)} F$ ($\mathrm{dom}(i)\colon \mathbf{J} \hookrightarrow \mathbf{J}'$),*

$$\frac{\mathrm{Lan}_{\mathrm{dom}(i)}F\colon \mathbf{J}' \to \mathrm{Zig}(C)}{\mathrm{Lan}_{\mathrm{Expl}(i)}(\textstyle\int F)\colon \mathrm{Expl}(\pi_C \circ \mathrm{Lan}_{\mathrm{dom}(i)}F) \to C}$$

In the case of colimits in $\mathrm{Zig}(C)$, this is tantamount to a collection of colimits in C.

Proof. The correspondence between left Kan extensions is immediate from Theorem 2.10 with the observation that Zig is a parametric right adjoint.

For the subsequent claim, we view a colimit of $F\colon \mathbf{J} \to \mathrm{Zig}(C)$ as in Proposition 2.3. $\mathbf{J}^{\triangleright}$ as a Δ_+-indexed category is completely determined from F, except at the terminal object $1 \in \mathbf{J}^{\triangleright}$; however, using step 1 (Section 2.1.1), we compute the shape of the colimit zigzag (if it exists), and hence the length of the exploded zigzag at this terminal object (along with the shape of accompanying maps into it). Thus, we realise i as a Δ_+-indexed functor and compute $\mathrm{Expl}(i)$. The resulting left Kan extension in C looks like a collection of colimit Kan extension diagrams glued together: one for each singular level of the exploded zigzag at the terminal object (glued along shared regular levels); each of these maximal sieves can be considered independently, as colimits. □

F.2 Proofs for Section 3 (Enriched categories of framed zigzags)

Lemma F.1. *For any poset P, every downwards-closed subset $S \in {\downarrow}P$ is the union of the principal downsets of its elements: $S = \bigcup_{d \in S} {\downarrow} d$.*

Lemma F.2 (Semilattice tensor product [9, 10]). *A map $h\colon X \times Y \to L$ is a $\vee$-bimorphism if it is $\vee$-preserving in each variable separately. $\bigvee\mathbf{Lat}$ is equipped with a symmetric monoidal structure given by the codomain of the universal $\vee$-bimorphism $X \times Y \to X \otimes Y$; i.e. any $\vee$-bimorphism h factors uniquely through a $\vee$-preserving map $\hat{h}\colon X \otimes Y \to L$. Explicitly, $X \otimes Y$ is the $\vee$-semilattice generated by formal elements $x \otimes y$ for $x \in X$ and $y \in Y$, subject to the relations making $\vee$-preserving maps out of $X \otimes Y$ the $\vee$-bimorphisms with domain $X \times Y$:*

$$(x_1 \vee x_2) \otimes y = (x_1 \otimes y) \vee (x_2 \otimes y), \quad x \otimes (y_1 \vee y_2) = (x \otimes y_1) \vee (x \otimes y_2).$$

The monoidal unit is the terminal $\vee$-semilattice 1.

Lemma 3.1. *The following are monoidal adjunctions:*

$$(\mathbf{Set}, \times, 1) \;\underset{U^{\leq}}{\overset{\Pi_0}{\underset{\perp}{\overset{\perp}{\overleftarrow{\xrightarrow{D}}}}}}\; (\mathbf{Pos}, \times, 1) \;\underset{U^{\vee}}{\overset{\downarrow}{\underset{\perp}{\rightleftarrows}}}\; (\textstyle\bigvee\mathbf{Lat}, \otimes, 1),$$

where $\otimes$ is the free semilattice tensor product (Lemma F.2), D is the discrete poset functor, $U^{\leq}$ is the underlying set functor, and Π_0 is the connected components functor.

Proof. For $\mathbf{Set}(\Pi_0(P), X) \cong \mathbf{Pos}(P, D(X))$, if given a function $f\colon \Pi_0(P) \to X$, construct a monotone map $\alpha\colon P \to D(X)$ as $p \mapsto f(\Pi_0(p))$; this is monotone because if $p \leq q$ the $\Pi_0(p) = \Pi_0(q)$. Conversely, given α, construct f as $[p] \mapsto \alpha(p)$; this function is well-defined because monotonicity for α ensures that if $p, q \in [p]$, then $\alpha(p) = \alpha(q)$ as $D(X)$ has the discrete order.

For $\mathbf{Pos}(D(X),P) \cong \mathbf{Set}(X,U^{\leq}(P))$, a monotone map $\alpha\colon D(X) \to P$ is precisely a function $X \to U^{\leq}(P)$ by discreteness of $D(X)$, and vice versa.

For $\bigvee\mathbf{Lat}(\downarrow P, L) \cong \mathbf{Pos}(P, U^{\vee}(L))$, given a $\vee$-preserving map $h : \downarrow P \to L$, construct a monotone map $\alpha : P \to U^{\vee}(L)$ as $p \mapsto h(\downarrow p)$; this is monotone because if $p \leq q$ then $\downarrow p \subseteq \downarrow q$ and hence $h(\downarrow p) \leq h(\downarrow q)$ by monotonicity of h. Conversely, given α, construct h as $S \mapsto \bigvee_{x \in S} \alpha(x)$, which is $\vee$-preserving and well-defined because S is nonempty.

Π_0 is strong monoidal because product projections preserve connected components and the product of connected componenets is connected, so it preserves product. D also preserves products (it is right adjoint). $\downarrow$ is strong monoidal because for posets P and Q, there is a canonical $\vee$-bimorphism $\downarrow P \times \downarrow Q \to \downarrow(P \times Q)$ given by $(S,S') \mapsto S \times S'$ (the product of nonempty downwards-closed sets is a nonempty downwards-closed set), so by the universal property of the tensor product this induces a unique $\vee$-preserving map $\downarrow P \otimes \downarrow Q \to \downarrow(P \times Q)$ which sends $S \otimes S' \mapsto S \times S'$. The inverse of this map is given by $S \mapsto \bigvee_{(p,q) \in S} \downarrow p \otimes \downarrow q$, and using Lemma F.1 one can check that these maps are mutually inverse. Thus, those adjunctions are monoidal. □

Lemma F.3 (Change of base [6]). *Let $N\colon \mathcal{V} \to \mathcal{W}$ be a lax monoidal functor. There is an induced 2-functor $N_*\colon \mathcal{V}\text{–}\mathbf{Cat} \to \mathcal{W}\text{–}\mathbf{Cat}$ called* change of base *along N which takes a $\mathcal{V}$-category $\underline{C}$ to a $\mathcal{W}$-category $N_*(\underline{C})$, with the same objects and Hom $\mathcal{W}$-objects given by $N_*(\underline{C})(X,Y) \coloneqq N(\underline{C}(X,Y))$.*

Moreover, N_* is an adjunction of 2-categories if and only if N is a monoidal adjunction, if and only if it is strong monoidal and admits a right adjoint as a functor.

Lemma F.4 (Sliced adjoint functors). *Let C be a category with pullbacks admitting an adjunction*

$$\mathcal{D} \underset{R}{\overset{L}{\underset{\perp}{\leftrightarrows}}} \mathcal{C}.$$

For every object $c \in C$, there is a corresponding adjunction between slice categories

$$\mathcal{D}/Lc \underset{R/_c}{\overset{L/_c}{\underset{\perp}{\leftrightarrows}}} \mathcal{C}/c,$$

where $L/_c$ is given by applying L and $R/_c \coloneqq \eta_c^* \circ R$ given by pullback along the unit of the adjunction $L \dashv R$ at c.

Lemma 3.2. *Define $\Pi_{0*}/_{D_*(\Delta_+)}\colon \mathbf{PosCat}/D_*(\Delta_+) \to \mathbf{Cat}/\Delta_+$ by applying Π_{0*}; then*

$$\begin{array}{ccc} \mathbf{Cat}\,/\,\boldsymbol{\Delta}_+ & \underset{D_*/\boldsymbol{\Delta}_+}{\overset{\Pi_{0*}/_{D_*(\boldsymbol{\Delta}_+)}}{\underset{\perp}{\leftrightarrows}}} & \mathbf{PosCat}\,/D_*(\boldsymbol{\Delta}_+) \\ {\scriptstyle\mathrm{dom}}\downarrow & & \downarrow{\scriptstyle\mathrm{dom}} \\ \mathbf{Cat} & \underset{D_*}{\overset{\Pi_{0*}}{\underset{\perp}{\leftrightarrows}}} & \mathbf{PosCat} \end{array}$$

commutes.

Proof. $\Pi_{0*}/_{D_*(\Delta_+)}$ is obtained as a sliced adjoint (Lemma F.4), left adjoint to $D_*/_{D_*(\Delta_+)}$. Now deduce that $D_*/_{D_*(\Delta_+)} = D_*/_{(\Delta_+)}$ because $\Pi_{0*} \circ D_* = \mathrm{id}_{\mathbf{Cat}}$ (because $\Pi_0 \circ D = \mathrm{id}_{\mathbf{Set}}$), hence the unit defining the pullback for $D_*/_{D_*(\Delta_+)}$ is the identity; now, a pullback along the identity gives the original morphism. □

Theorem 3.5 (Unframing). $\Pi_{0*}/_{D_*(\Delta_+)} \circ \underline{\mathrm{Zig}} = \mathrm{Zig} \circ \Pi_{0*}$.

Proof. Ultimately, this is because 2-cell fillers in framed zigzag **Pos**-categories are pointwise over their base. It suffices to check that $\Pi_{0*}/_{D_*(\Delta_+)}(\underline{\mathrm{Zig}(C)}) = \mathrm{Zig}(\Pi_{0*}(\underline{C}))$. Recall that

$$\mathrm{Zig}(\Pi_{0*}(\underline{C})) \coloneqq U_*^{\leq}/_{D_*(\Delta_+)}(\underline{\mathrm{Zig}}(D_*(\Pi_{0*}(\underline{C})))).$$

Both categories have the same objects, because every slice/change of base functor is the identity on objects, so it suffices to check the morphisms. For the left, the morphisms are given by connected components of framed zigzag maps. Conversely, on the right, the morphisms are given by zigzag maps whose type are given by connected components of $\underline{C}$-morphisms. These are identical, because $\underline{\mathrm{Zig}(C)}$ inherits its order pointwise from $\underline{C}$: a pair of framed zigzag maps are in the same connected component if and only if for every pair of corresponding components, the $\underline{C}$-morphisms are in the same connected component. □

Lemma 3.3. *If F admits a marked colimit $\mathrm{colim}^{\mathbf{M}} F$ for any $\mathbf{M}$, $\Pi_{0*}(F)\colon \Pi_{0*}(D(C(J))) \to \Pi_{0*}(\underline{C})$ admits a colimit given by the same object whose legs are given by applying Π_{0*}.*

Proof. Π_{0*} sends oplaxly commuting diagrams to commuting diagrams, and hence any marked cocone (regardless of marking) to a cocone. □