homotopy.io

a proof assistant for finitely-presented globular n-categories

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Structure of this talk

- Background information on monoidal categories
- Whirlwind tour of the foundations of homotopy.io
- String diagram success stories
- homotopy.io demo

Monoidal categories

Ordinary categories I

- Ordinary categories are the theory of *sequential* composition
- Unlike set theory, rather than focusing on what mathematical objects *are*, focus on how they *interact* (via composition)
  - Yoneda lemma

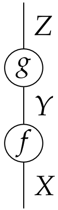
  You work at a particle accelerator. You want to understand some particle. All you can do are throw other particles at it and see what happens. If you understand how your mystery particle responds to all possible test particles at all possible test energies, then you know everything there is to know about your mystery particle. Vakil 2009

Ordinary categories II

- Admits a graphical calculus:

$$(Y \xrightarrow{g} Z) \circ (X \xrightarrow{f} Y) = X \xrightarrow{f} Y \xrightarrow{g} Z$$

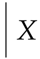
- String diagrams:



Ordinary categories III

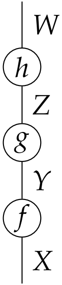
- Identity law justifies representing objects as wires:

$$X \xrightarrow{id(X)} X \xrightarrow{f} Y = X \xrightarrow{f} Y = X \xrightarrow{f} Y \xrightarrow{id(Y)} Y \quad \rightsquigarrow$$



- Associativity justifies lack of bracketing:

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \xrightarrow{h} W = X \xrightarrow{f} (Y \xrightarrow{g} Z \xrightarrow{h} W) \quad \rightsquigarrow$$



Monoidal categories

Definition

A monoidal category is a category $\mathscr{C}$ equipped with

- a tensor product functor $\mathscr{C} \times \mathscr{C} \xrightarrow{\otimes} \mathscr{C}$,
- a distinguished (unit) object I,
- an associator natural isomorphism $(- \otimes -) \otimes - \overset{\alpha}{\cong} - \otimes (- \otimes -)$,
- left and right unitor natural isomorphisms $I \otimes - \overset{\lambda}{\cong} -$, $- \otimes I \overset{\lambda}{\cong} -$,

satisfying the triangle equation:

$$(X \otimes I) \otimes Y \xrightarrow{\alpha_{X,I,Y}} X \otimes (I \otimes Y)$$

$$(X \otimes I) \otimes Y \xrightarrow{\rho_X \otimes id(Y)} X \otimes Y \xleftarrow{id(X) \otimes \lambda_Y} X \otimes (I \otimes Y)$$

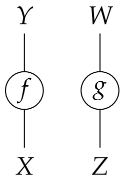
and pentagon equation:

$$((X \otimes Y) \otimes Z) \otimes W \xrightarrow{\alpha_{X,Y,Z} \otimes id(W)} (X \otimes (Y \otimes Z)) \otimes W \xrightarrow{\alpha_{X,Y \otimes Z,W}} X \otimes ((Y \otimes Z) \otimes W) \xrightarrow{id(X) \otimes \alpha_{Y,Z,W}} X \otimes (Y \otimes (Z$$

$$((X \otimes Y) \otimes Z) \otimes W \xrightarrow{\alpha_{X \otimes Y,Z,W}} (X \otimes Y) \otimes (Z \otimes W) \xrightarrow{\alpha_{X,Y,Z \otimes W}} X \otimes (Y \otimes (Z$$

Systems and processes

- Monoidal categories are the theory of *parallel* composition
- Interpret the objects (X, Y, Z, ...) of a category as *systems*, and morphisms (f, g, ...) as *processes*
  - Then the $\otimes$-composite $f \otimes g$ is then the process where f and g 'happen at the same time'



Coherence

- The category $\mathbf{Set}$ is monoidal with $\otimes := \times$
  - ... but we don't have $A \times (B \times C) = (A \times B) \times C$
  - ... but we do have a (natural!) isomorphism $A \times (B \times C) \overset{\alpha}{\cong} (A \times B) \times C$
  - Similarly, we don't have $\{*\} \times A = A$ *on-the-nose*, but an obvious canonical isomorphism $\{*\} \times A \overset{\lambda}{\cong} A$

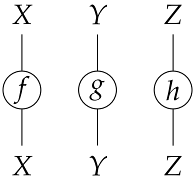
Theorem (Coherence for monoidal categories (Lane 1963))

In a monoidal category, any pair of parallel morphisms built from α, λ, and ρ, are equal.

- Coherence morphisms only encode structural and 'bureaucratic' information

String diagrams

- Idea: quotient by coherence, and use topology to represent the structural information
- Given $X \xrightarrow{f} X$, $Y \xrightarrow{g} Y$, and $Z \xrightarrow{h} Z$, draw **both** $(X \otimes Y) \otimes Z \xrightarrow{(f \otimes g) \otimes h} (X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z) \xrightarrow{f \otimes (g \otimes h)} X \otimes (Y \otimes Z)$ as:



Theorem (Correctness of graphical calculus for monoidal categories (Joyal and Street 1991))

A well-typed equation between morphisms in a monoidal category holds automatically if and only if the corresponding string diagrams are planar isotopic.

Where do coherence results come from?

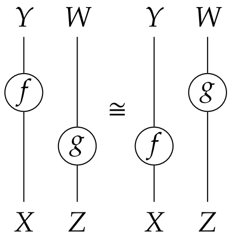
Theorem (Strictification for monoidal categories)

Every monoidal category is monoidally equivalent to a strict one.

- Analogous results hold for braided and symmetric monoidal categories
- Every (weak) 2-category is equivalent to a strict 2-category...
- ...but **not** every (weak) 3-category is equivalent to a strict one!
- Gray categories give a semistrict, fully-expressive, description of weak 3-categories with strict associators, strict unitors, and *weak interchangers* (Gordon, Power and Street 1995)
- Associative n-categories (Dorn 2018) agree with Gray categories in dimension 3, and are conjectured to be equivalent to weak n-categories

Weak interchange

In a monoidal category, we have a planar isotopy:



The witness is called the interchanger; strict interchange makes this an equation

Multiplicity versus dimensionality

Proposition

A monoidal category is precisely a 2-category which has only a single object.

Analogous to the following:

Proposition

A monoid is precisely a 1-category which has only a single object.

- When a 2-category has many objects, this is represented with coloured regions in string diagrams
- Key idea: n-categories are most general, and correspond to n-ways to compose

Periodic table

k\n	0	1	2	...
0	sets	categories	2-categories	...
1	monoids	monoidal categories	monoidal 2-categories	...
2	commutative monoids	braided monoidal categories	braided monoidal 2-categories	...
3		symmetric monoidal categories	sylleptic monoidal 2-categories	...
4			symmetric monoidal 2-categories	...
$\vdots$				$\ddots$

(Baez and Dolan 1995)

n-categories

Definition (Globular weak higher categories)

A weak $(n+1)$-category is a category weakly-enriched in n-categories, where $\mathbf{Set}$ is the category of 0-categories.

- Simple intuitively, but notoriously hard to wrangle...
- Necessary coherence data explodes exponentially with dimension
- Difficulty lies in obtaining something weak enough to be expressive, but strict enough to be usable: there is no clear 'best' semistrict theory! (Cheng and Lauda 2004)
- Our approach: n-dimensional string diagrams are well-behaved; derive a semistrict presentation from that

Zigzags I

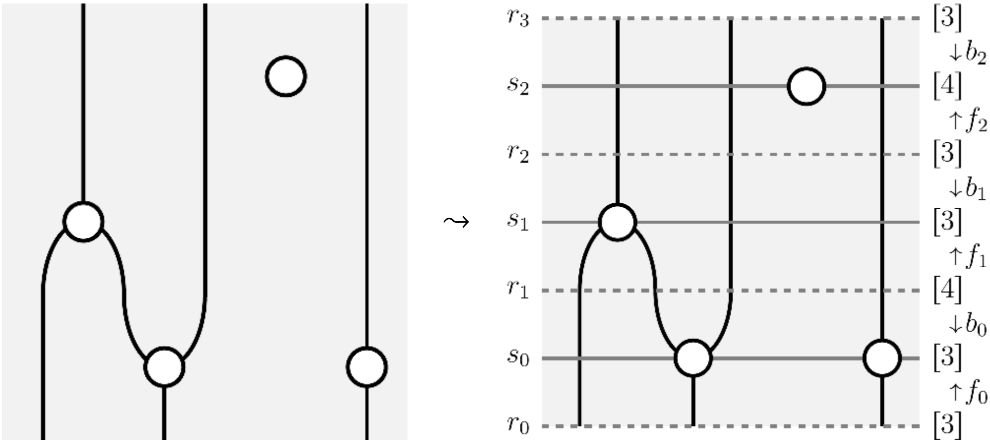


Figure 1: Encoding a 2D string diagram (Reutter and Vicary 2019)

Zigzags II

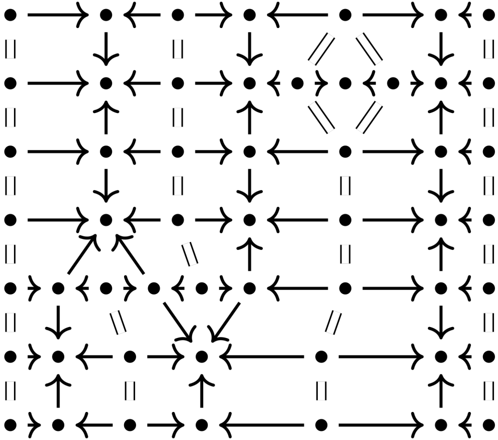


Figure 2: Figure 1 wireframe, with explicit rewrites

Zigzags III

Definition (Zigzag map)

In a category $\mathscr{C}$, a zigzag map between two iterated cospans (zigzags) is given by a monotone function between apexes and $\mathscr{C}$-morphisms for each apex in the source, subject to some conditions.

Definition (Zigzag category)

The zigzag category $Z_{\mathscr{C}}$ has as objects zigzags in $\mathscr{C}$ and zigzag maps as morphisms.

- $Z_{\mathbf{1}}$ is the simplex category Δ; i.e. the category of non-empty finite ordinals and monotone maps
- $Z_{\mathbf{1}}^2$ has objects *shapes* of 2D string diagrams
- $Z_{\mathbf{1}}^n$ has objects *shapes* of n-dimensional string diagrams

Zigzag contraction

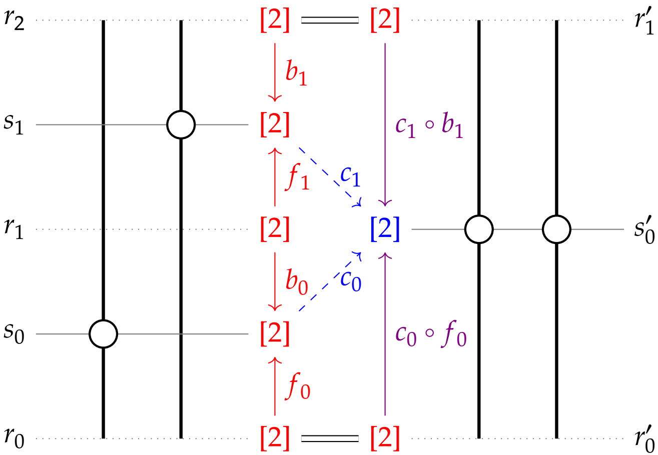
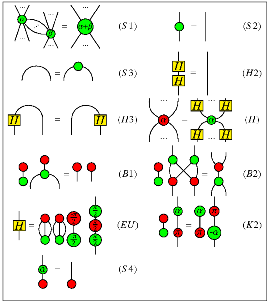


Figure 3: Contracting the two-bead diagram

Applications

Categorical Quantum Mechanics

- Traditionally, quantum information theory happens in the category **FHilb** of finite-dimensional Hilbert spaces
- **FHilb** is dagger-compact, and many constructions generalise to this setting
- In particular, **Rel** is also dagger-compact, and acts as a simplified ‘half-way’ point between classical physics (embodied by **Set**) and quantum physics (**FHilb**)
- Success story: ZX-calculus (Hadzihasanovic, Ng and Wang 2018)



Diagrammatic algebra

- Petri nets provide a formalism for dynamical systems, and are intrinsically related to symmetric monoidal categories
- Linear algebra can be done graphically, using the graphical calculus of string diagrams à la generators-and-relations (Bonchi, Sobocinski and Zanasi 2017)
- Bonchi, Holland et al. (2019) define a resource calculus with string diagrams, which generalise signal flow graphs

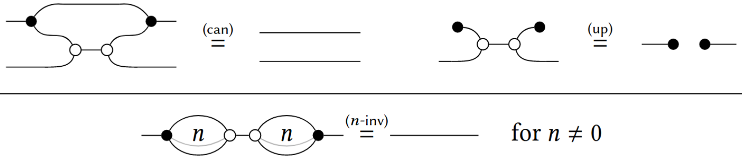


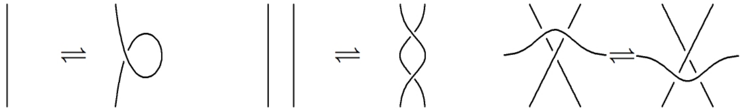
Figure 5: Diagrammatic algebra

Knot theory

- Braided monoidal categories are the setting for knot theory

Theorem

Two knots are isotopic in 3D if and only if their 2D projections are equivalent by Reidemeister moves:



(Diagrams from (Armstrong 2007))

- Our system intrinsically knows about the second and third Reidemeister moves!

Sphere eversion

- Outside In — How to turn a sphere inside out

Theorem
S^0, S^2, and S^6 are the only spheres which admit eversion.

- No hope of writing out the S^6 eversion graphically, until now...

Further work

- Homotopy type theory (HoTT)
  - HoTT is very good at *logical* constructions, but leaves a lot to be desired for *topological* arguments
  - Syllepsis was only proven last year (Sojakova 2021)
  - homotopy.io is the opposite — ideally, we want to take a proof in homotopy.io and automatically translate it into HoTT; we can leverage CaTT (Finster and Mimram 2017):

$$\text{homotopy.io} \rightsquigarrow \text{CaTT} \rightsquigarrow \text{HoTT}$$

- homotopy.io is based on the theory of Associative n-categories
  - By 'homotopy', we mean something entirely combinatorial rather than topological
  - Need something like a model structure for this to be made precise
- How to handle n-groupoids instead of n-categories?
  - We require a mechanism to generate *coherent* inverses for each element in our signature

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