PROFUNCTOR STRING DIAGRAMS

Nick Hu

What is a string diagram? I

- Monad axioms, for a

  - monad $\mathcal{C} \xrightarrow{T} \mathcal{C}$

  - unit
$$\begin{tikzcd} \mathcal{C} \arrow[rr, equal, "Id"] \arrow[rr, bend right=60, "T"'] & \arrow[phantom, "\Downarrow\eta"] & \mathcal{C} \end{tikzcd}$$

  - multiplication
$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\ \mathcal{C} \arrow[ur, "T"] \arrow[rr, "T"'] & {} & \mathcal{C} \end{tikzcd}$$

"associativity"
$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "\mu T"] \arrow[d, Rightarrow, "T\mu"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd}$$

"unit"
$$\begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

Poincaré Duality

Pasting diagrams	String diagrams
Points	Surfaces
Lines	Lines
Surfaces	Points

in 2D

Pasting diagrams	String diagrams
Points	Volumes
Lines	Surfaces
Surfaces	Lines
Volumes	Points

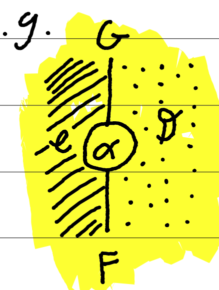
in 3D

- Generally, n-dimensional things in k-dimensional space become $(k-n)$-dimensional, e.g.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$$

Poincaré dual

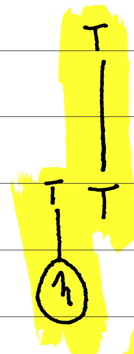
$\longleftrightarrow$



What is a string diagram? II

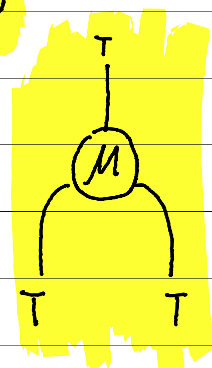
· Monad axioms, for a

• monad

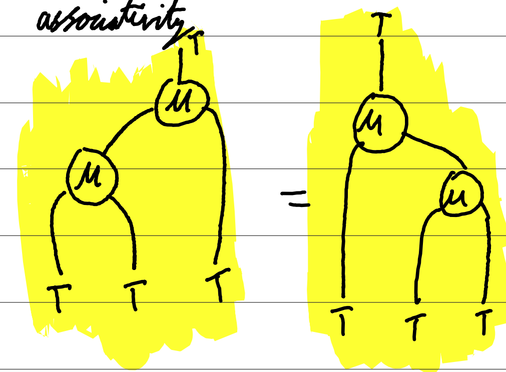


• unit

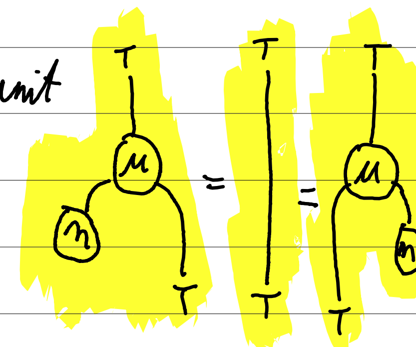
• multiplication



associativity



unit

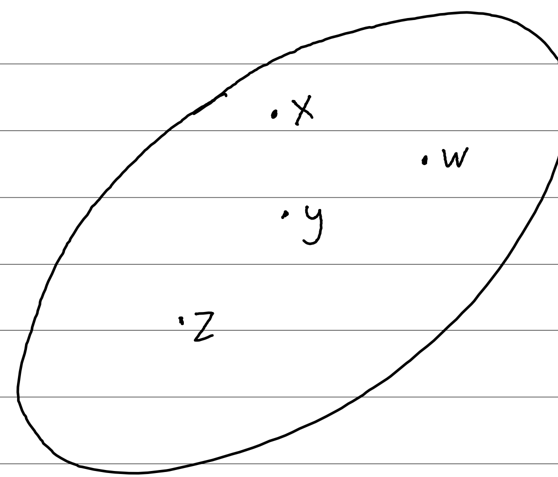


String diagrams for Sets

- A set is just a 0-category

<u>Definition</u> Category:

- A set of objects (0-morphisms)
- For each pair of objects, a set of (1-) morphisms
- A unital and associative binary composition operation on (1-) morphisms

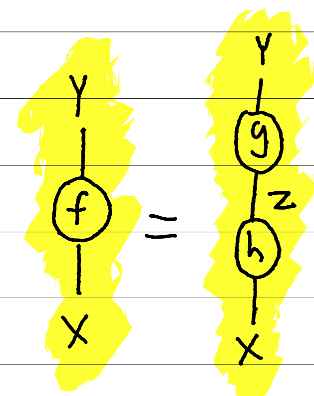


String diagrams for (ordinary) Categories

Definition Category:

- A set of objects (0-morphisms)
- For each pair of objects, a set of (1-) morphisms
- A unital and associative binary composition operation on (1-) morphisms

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \\ Z \arrow[ur, "g"'] & \end{tikzcd}$$



2D diagrams

- What's so special about monoidal categories?

<u>Theorem</u>. Delooping
Every monoidal category is equivalent to a one-object bicategory (weak 2-category).

... really just a generalisation of the idea that every monoid is equivalent to a one-object category

Diagrams in n dimensions

A n-category is one which has n composition operations. Each composition corresponds to a diagrammatic dimension.

Definition n-Category:

- A set of objects (0-morphisms)
- For each pair of objects, a set of (1-) morphisms
- A unital and associative binary composition operation on (1-) morphisms

First composition

- For each pair of parallel (1-) morphisms, a set of 2-morphisms
- 2-morphisms compose subject to triangle and pentagon equations

Second composition

- ... 3-morphisms ...
- ... 3-morphism coherence ...
- .....

Profunctors I

External analysis of a fixed category $\mathcal{C}$:

- In Cat, one can recover any object of $\mathcal{C}$ by choosing an appropriate functor

$$1 \xrightarrow{x} \mathcal{C}$$

... but what about morphisms?

Profunctors II

<u>Definition</u> Profunctor.
A profunctor $\mathcal{C} \nrightarrow \mathcal{D}$ is a $\mathcal{D}^{op} \times \mathcal{C} \to \mathrm{Set}$ functor.

- Hom is a profunctor!

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\mathcal{C}(-,=)} \mathrm{Set}$$

- Prof is the compact closed bicategory of
  0: categories
  1: profunctors
  2: natural transformations

- The identity profunctor, cups, and caps are all Hom.

Profunctors III

- When are functors and profunctors the same?
  - A $\mathcal{C} \nrightarrow \mathcal{D}$ profunctor is the same as a $\mathcal{C} \to \mathcal{D}$ functor when it admits a right adjoint.*

- There are two embeddings $Cat \to Prof$, one covariant and one contravariant, which admit an adjunction:

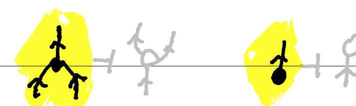
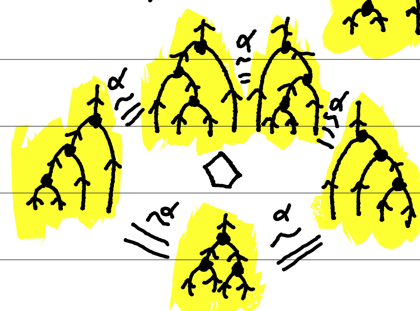
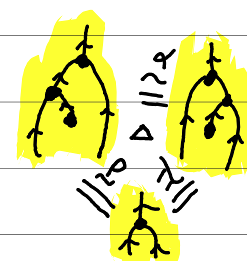
$$\begin{tikzcd} Cat \arrow[r, bend left, "(-)_*"] \arrow[r, bend right, "(-)^*"'] & Prof \end{tikzcd}$$

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \longmapsto \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_*", "\shortmid" marking] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "F^*", "\shortmid" marking] \end{tikzcd}$$

Monoidal categories, externally

$$\begin{tikzcd} \mathrm{Free}(P) \arrow[r, "Z"] \arrow[dr, "\tilde{Z}"'] & \mathrm{Cat} \arrow[d, "(-)_\bullet"] \arrow[dl, phantom, "\Leftarrow \xi"] \\ & \mathrm{Prof} \end{tikzcd}$$

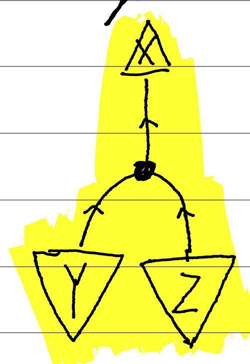
Consider the following presentation P:

- a generating 0-morphism: $*$
- generating 1-morphisms: 
- invertible generating 2-morphisms: 
- coherence equations  

Internal string diagrams

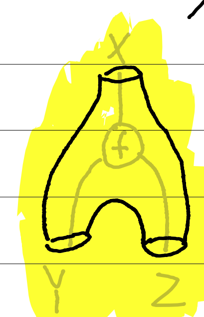
- In a monoidal category $\mathcal{C}$, we have a diagrammatic calculus, e.g. $X \xrightarrow{f} Y \otimes Z$ as

- In the diagrammatic calculus of Prof



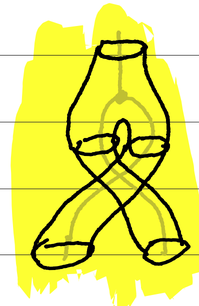
corresponds to $\mathcal{C}(X, Y \otimes Z)$

- $f \in \mathcal{C}(X, Y \otimes Z)$, so draw like this:

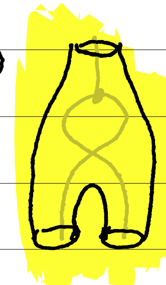


Examples

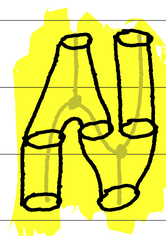
- Symmetry



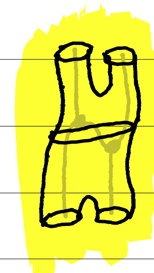
$\xmapsto[\sim]{\tilde{Z}(\sigma)}$



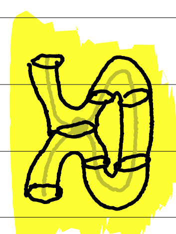
- Compactness



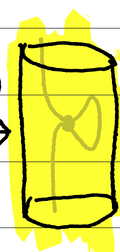
$\xmapsto[\sim]{\tilde{Z}(\theta)}$



- Tracedness



$\xmapsto{\tilde{Z}(\mathrm{Tr})}$



Thanks for listening!